# ACM 204, FALL 2018: LECTURES ON CONVEX GEOMETRY

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Caltech CMS Lecture Notes No. 2019-02
doi:10.7907/geda-h205

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Caltech / ACM 204 / Fall 2018:

# Lectures on Convex Geometry

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*Typeset on December 19, 2018*

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**In Fall 2018,** I took a stab at developing a one-quarter survey class on convex geometry for an audience of second- and third-year graduate students at Caltech. These lecture notes document the first version of the course, as it was actually given.

Unfortunately, 10 weeks do not suffice to reach the frontiers of a field that has been explored for over 100 years. Therefore, the mission for the class was to guide the students to some of the peaks in the theory of convex geometry so that they could begin to see the beauty and extent of this territory.

I chose to cover topics in pure geometry that appear surprising and wonderful, without much concern for the immediate applications. The material includes some of the most classical topics, such as the Euler–Poincaré–Schläfli formula, Steiner's formula, and the isoperimetric inequality. But it also reaches toward more modern perspectives, such as Barthe's proof of the geometric Brascamp–Lieb inequality via optimal transportation. We finished the course with a new proof of the Alexandrov–Fenchel inequality that appeared on arXiv last month (!). The students will judge whether this synthesis was successful.

There is no pretense that the course gives a comprehensive treatment of any part of convex geometry. Even subject to this limitation, there are a number of ancillary topics that are missing. For instance, we did not discuss conic geometry, smooth convex bodies, hyperbolic polynomials and polarization, mixed discriminants, monotone transportation, log-concave distributions and measures, almost-spherical slices of convex bodies, ideas from asymptotic convex geometry, the connections with information theory, or any applications. At some point, I may reform and extend these lectures to address these shortcomings.

Right now, these notes have some severe deficiencies. They repeat the material as it was presented, without the benefit of review or reflection. Many of the lectures contain my first attempt to explain a subject, and the results are not always very satisfying. The notes are almost totally void of citations to the sources that I used to prepare the lectures, and they lack most of the background and context that would appear in a published work. Although I have edited the lectures for English usage and (somewhat) consistent typography, I did not check or correct the mathematical content. Caveat lector!

I would like to thank the participants in this course for their interest and attention. They are Ameera Abdelaziz, Jeremy Bernstein, Jialin Chen, Elijah Cole, Victor Dorobantu, Spencer Gordon, SooJean Han, Dimitar Ho, De Huang, Richard Kueng, Tongxin Li, Riley Murray, Florian Schaefer, Andrew J. Taylor, Oguzhan Teke, Lucian Werner, Recep Can Yavas, Jing Yu, Fangzhou Xiao, and Shumao Zhang. My former student Michael B. McCoy wrote the LaTeX template that supports these lecture notes. Without their effort, there would be no typed lecture notes, no figures, and no homework solutions. Nevertheless, any errors are my sole responsibility.

I would also like to thank Sasha Barvinok, whose courses introduced me to the deeper mysteries of convex geometry. My enduring interest stems from my research collaboration with Dennis Amelunxen, Martin Lotz, and Michael McCoy. Finally, I am grateful to Ramon Van Handel for extensive discussions about this class; in particular, he sent me a preliminary version of his manuscript on the Alexandrov–Fenchel inequality so that I could present it.

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December 2018

# Lecture 1: Convex sets from the inside

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*ACM 204, Fall 2018*
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*2 October 2018*

## 1.1 Agenda for Lecture 1

This lecture gives some context for the course and some background information about affine geometry. We continue with the definition of a convex set, the central object in this course. The main focus of today's lecture is the internal structure of a convex set. The key idea is that a convex set contains all (finite) averages of points drawn from the convex set. We also single out special subsets, called faces, whose points *cannot* be written as averages of points in the set from outside the face.

1. Overview and history
2. Affine geometry
3. Convex sets and convex hulls
4. The Carathéodory theorem
5. Basic topology of convex sets
6. Extreme points and faces

## 1.2 Introduction

ACM/IDS 204 is a topics course on convexity. This term, the focus is on Brunn–Minkowski theory. Roughly, this subfield of convex geometry concerns the interaction between convexity and volume in finite dimensions.

Brunn–Minkowski theory is named after two founders of the formal study of convexity. Karl Hermann Brunn (1862–1939) was a German mathematician who worked on convexity and knot theory. His 1887 thesis, *Über Ovale und Eiflächen*, is one of the earliest general treatments of convex bodies. It contains a well-known result called the Brunn slicing theorem, which we will prove.

Hermann Minkowski (1864–1909) was born in what is now Kaunas, Lithuania; he worked in Germany and Switzerland. Minkowski laid the foundations of convexity theory, including some of the most basic definitions and results. He was particularly interested in interactions between lattices and convex bodies, which blossomed into a field called the geometry of numbers. Minkowski is also known for his work on the four-dimensional geometry of space–time.

The names of Brunn and Minkowski are united in the famous Brunn–Minkowski theorem, which asserts that the volume is a log-concave function. We will state and prove the modern version of their result later this term.

Convexity, as a subject, has its roots in the earliest geometrical investigations. Polytopes, polyhedra, and dissections appear already in Euclid's *Elements*. The regular convex bodies (the tetrahedron, cube, octahedron, dodecahedron, and icosahedron) were also known to the Greeks. The isoperimetric problem in the plane, which we will discuss, is associated with a legendary story about Dido, the queen of Carthage and a supporting character in Vergil's *Aeneid*.

Before the systematic work of Brunn & Minkowski, there were other substantial investigations into the geometry of regular figures in the plane and space. Kepler (1571–1630) proved that there are only 13 semiregular Archimedean solids, and he studied a problem that we now call sphere packing. Euler (1707–1783) established the first result on combinatorial topology, the relation among the vertices, edges, and faces of a convex polytope in space; we will prove the modern version later this term. Other important 19th century researchers who worked on convex geometry include Gauss (1777–1855; constructible polygons, curvature, lattice packings, non-Euclidean geometry), Steiner (1796–1863; expansion, symmetrization, isoperimetry), Cauchy (1798–1857; rigidity, surface area formula), and Schläfli (1814–1895; higher-dimensional polytopes).

For a summary of the history of convexity theory, see [Gru93]. We may discuss more of this background as it arises.

This course is *sui generis*. We will draw material from a wide range of sources, including the papers and books listed on the syllabus. Three particularly useful references are Barvinok's textbook [Bar02], Gruber's graduate-level survey [Gru07], and Schneider's *magnum opus* [Sch14].

## 1.3 Setting and Notation

Convexity is a subject that takes place in a *real* vector space. This term, we will be working in the simplest class of real vectors spaces, namely the finite-dimensional Euclidean spaces.

### 1.3.1 Linear Structure

Recall that $\mathbb{R}^d$ is the family of all $d$-dimensional (column) vectors with real entries:

$$\mathbb{R}^d := \{(x_1, \dots, x_d) : x_i \in \mathbb{R} \text{ for } i = 1, \dots, d.\}.$$

We make $\mathbb{R}^d$ a linear space in the usual way, equipping it with scalar multiplication and addition, performed coordinate by coordinate:

$$\alpha \cdot (x_1, \dots, x_d) := (\alpha x_1, \dots, \alpha x_d) \quad \text{for } \alpha \in \mathbb{R};$$
$$(x_1, \dots, x_d) + (y_1, \dots, y_d) := (x_1 + y_1, \dots, x_d + y_d).$$

We use lowercase Greek letters $(\alpha, \beta, \lambda, \tau)$ for scalars. Boldface italic letters $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ refer to vectors in $\mathbb{R}^d$, and we use subscripted italic letters $(x_i, y_i, z_i)$ to refer to individual coordinates of the vectors. We write $\mathbf{0} := (0, \dots, 0) \in \mathbb{R}^d$ for the zero vector, also known as the *origin*.

It is sometimes convenient to work with the standard basis for $\mathbb{R}^d$. This basis consists of the vectors that have a unit entry in a specific coordinate and zeros elsewhere:

$$\mathbf{e}_i := (\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0) \in \mathbb{R}^d \quad \text{for } i = 1, \dots, d.$$

We sometimes write $\mathbf{e} := (1, \dots, 1) \in \mathbb{R}^d$ for the vector of ones.

### 1.3.2 Geometry

Next, let us add geometry in the form of angles and distances. We introduce the standard inner product, which reflects the "similarity" between two vectors:

$$\langle \boldsymbol{x},\, \boldsymbol{y} \rangle := \sum\nolimits_{i=1}^d x_i y_i \quad \text{for } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d.$$

The inner product induces the Euclidean norm, which reflects the magnitude of a vector:

$$\|\boldsymbol{x}\|^2 := \langle \boldsymbol{x},\, \boldsymbol{x} \rangle = \sum\nolimits_{i=1}^{d} x_i^2 \quad \text{for } \boldsymbol{x} \in \mathbb{R}^d.$$

The Cauchy–Schwarz inequality provides a bound on the inner product:

$$|\langle \boldsymbol{x},\, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\| \quad \text{for } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d.$$

We can now define the cosine of the angle between a pair of vectors:

$$\cos\theta(\boldsymbol{x}, \boldsymbol{y}) := \frac{\langle \boldsymbol{x},\, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\| \|\boldsymbol{y}\|} \quad \text{for } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d.$$

It is convenient to parameterize angles in the range $[0, 2\pi)$.

### 1.3.3 Topology

The Euclidean norm also induces a topology on $\mathbb{R}^d$. For a sequence $\{\boldsymbol{x}_k : k \in \mathbb{N}\} \subset \mathbb{R}^d$ and a point $\boldsymbol{x} \in \mathbb{R}^d$,

$$\boldsymbol{x}_k \to \boldsymbol{x} \quad \text{if and only if} \quad \|\boldsymbol{x}_k - \boldsymbol{x}\| \to 0.$$

The basic open sets (i.e., neighborhoods) and closed sets in the norm topology are just the open and closed norm balls. For $\boldsymbol{x} \in \mathbb{R}^d$ and $r \geq 0$, we define open and closed balls:

$$\mathsf{N}(\boldsymbol{x}; r) := \{\boldsymbol{y} \in \mathbb{R}^d : \|\boldsymbol{y} - \boldsymbol{x}\| < r\};$$
$$\bar{\mathsf{N}}(\boldsymbol{x}; r) := \{\boldsymbol{y} \in \mathbb{R}^d : \|\boldsymbol{y} - \boldsymbol{x}\| \leq r\}.$$

The empty set $\emptyset$ and the space $\mathbb{R}^d$ are both open and closed. We use sans serif capitals $(\mathsf{A}, \mathsf{B}, \mathsf{C})$ to denote sets.

Let $\mathsf{A} \subset \mathbb{R}^d$. An *interior point* of a set is one that is contained in an open subset of the set, and the *interior* $\operatorname{int} \mathsf{A}$ is the family of all interior points of the set:

$$\boldsymbol{x} \in \operatorname{int} \mathsf{A} \quad \text{if and only if} \quad \mathsf{N}(\boldsymbol{x}; r) \subset \mathsf{A} \quad \text{for some } r > 0.$$

A *limit point* of the set is the limit of a convergent sequence of points drawn from the set, and the *closure* $\operatorname{cl} \mathsf{A}$ is the family of all limit points of the set:

$$\boldsymbol{x} \in \operatorname{cl} \mathsf{A} \quad \text{if and only if} \quad \boldsymbol{x}_k \to \boldsymbol{x} \quad \text{where } \boldsymbol{x}_k \in \mathsf{A} \text{ for } k \in \mathbb{N}.$$

The *boundary* $\operatorname{bd} \mathsf{A}$ of the set consists of points in the closure that are not in the interior: $\operatorname{bd} \mathsf{A} := \operatorname{cl} \mathsf{A} \setminus \operatorname{int} \mathsf{A}$.

A set is *closed* if it coincides with its closure; that is, a closed set contains all its limit points. A set is *bounded* if it is contained in a closed ball about the origin:

$$\mathsf{A} \text{ is bounded} \quad \text{if and only if} \quad \mathsf{A} \subset \bar{\mathsf{N}}(\mathbf{0}; r) \quad \text{for some (finite) } r.$$

A set is *compact* if and only if every sequence from the set contains a subsequence that converges in the set. More precisely, for any sequence $\{\boldsymbol{x}_k : k \in \mathbb{N}\} \subset \mathsf{A}$, there are indices $k_\ell$ for $\ell \in \mathbb{N}$ so that the sequence $\boldsymbol{x}_{k_\ell}$ converges to a point $\boldsymbol{x} \in \mathsf{A}$ as the subindex $\ell \to \infty$. The Heine–Borel theorem asserts that, in $\mathbb{R}^d$, a set is compact if and only if it is closed and bounded.

We conclude with a technicality. The trivial linear space $\mathbb{R}^0 := \{\mathbf{0}\}$ consists of the origin only. The singleton $\{\mathbf{0}\}$ is both an open and a closed set. It is convenient to regard the origin as an interior point of $\mathbb{R}^0$ and to assert that $\mathbb{R}^0$ has no boundary points.

Warning 1.3.1 (Subsets). In these lectures, the symbol $\subset$ denotes an arbitrary subset. This notation does not exclude the possibility of equality. We write $\subsetneq$ to emphasize that the subset is proper.

Warning 1.3.2 (Compactness). Here is the general definition of compactness. A subset of a topological space is compact if and only if every open cover of the subset has a finite subcover. If we specialize to $\mathbb{R}^d$, this definition states that a set $\mathsf{A} \subset \mathbb{R}^d$ is compact when the following condition holds. For an arbitrary index set $I$, points $\boldsymbol{x}_i \in \mathbb{R}^d$, and numbers $r_i \in \mathbb{R}$,

$$\mathsf{A} \subset \bigcup\nolimits_{i \in I} \mathsf{N}(\boldsymbol{x}_i; r_i) \quad \text{implies} \quad \mathsf{A} \subset \bigcup\nolimits_{j \in J} \mathsf{N}(\boldsymbol{x}_j; r_j) \quad \text{for some finite subset } J \text{ of } I.$$

This general definition can be useful when we undertake packing and covering arguments. Although compactness and sequential compactness are equivalent in a metric space (such as a normed linear space), they are not equivalent in general.

The Heine–Borel equivalence (closed + bounded = compact) also fails in general; indeed, it does not hold in any infinite-dimensional normed linear space.

### 1.3.4 Volume

For a (Lebesgue) measurable set $\mathsf{A} \subset \mathbb{R}^d$, the *volume* $\mathrm{Vol}_d(\mathsf{A})$ is the Lebesgue content of the set. To be more precise, we introduce the (ordinary) indicator function

$$\mathbb{1}_{\mathsf{A}}(\boldsymbol{x}) := \begin{cases} 1, & \boldsymbol{x} \in \mathsf{A} \\ 0, & \boldsymbol{x} \notin \mathsf{A}. \end{cases}$$

Writing $\mathrm{d}\boldsymbol{x}$ for the Lebesgue measure,

$$\mathrm{Vol}_d(\mathsf{A}) := \int \mathbb{1}_{\mathsf{A}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

The 0-dimensional volume is defined by the relations

$$\mathrm{Vol}_0(\{\mathbf{0}\}) = 1 \quad \text{and} \quad \mathrm{Vol}_0(\emptyset) = 0.$$

If the dimension is clear, we may also drop the subscript and write simply $\mathrm{Vol}(\mathsf{A})$.

For the most part, measure theory will not play a role in this course. Indeed, for a convex set $\mathsf{C}$ in $\mathbb{R}^d$, the Lebesgue volume $\mathrm{Vol}_d(\mathsf{C})$ is equivalent with more elementary notions of volume. For example, we can approximate $\mathsf{C}$ by a disjoint family of small cubes, total the volumes of the cubes, and take the limit as the cubes grow smaller.

## 1.4 Affine Geometry

We assume that you, the student, are familiar with linear algebra and basic notions of linear geometry. Core ideas include linear subspaces, linear hulls, linear independence, and linear maps. It is less common for introductory classes to treat affine geometry, so we begin with a reminder of the essential concepts from this subject.

### 1.4.1 Affine Spaces

As you well know, two *distinct* points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ span a unique line:

$$\operatorname{line}(\boldsymbol{x}, \boldsymbol{y}) := \{(1-\alpha)\boldsymbol{x} + \alpha\boldsymbol{y} : \alpha \in \mathbb{R}\}. \tag{1.4.1}$$

A line does not need to contain the origin, but it is parallel with a unique line that does contain the origin. Note that the definition (1.4.1) remains valid when $\boldsymbol{x} = \boldsymbol{y}$, in which case the line collapses to a single point.

The notion of a line extends to the notion of an affine space.

**Definition 1.4.1** (Affine space)**.** A set $\mathsf{L} \subset \mathbb{R}^d$ is called an *affine space* if it contains all of its lines:

$$\boldsymbol{x}, \boldsymbol{y} \in \mathsf{L} \quad \text{implies that} \quad \operatorname{line}(\boldsymbol{x}, \boldsymbol{y}) \subset \mathsf{L}.$$

The empty set $\emptyset$ is (vacuously) affine. An affine space is also called a *flat*.

Each affine space is parallel with a unique linear subspace (that contains the origin), and we can define the dimension of an affine space to equal the dimension of the parallel linear subspace. Examples of affine spaces include a single point (with dimension zero), a line (with dimension one), a plane (with dimension two), a hyperplane (with dimension $d-1$), and the entirety of $\mathbb{R}^d$ (with dimension $d$).

Each affine space $\mathsf{L} \subset \mathbb{R}^d$ is a closed subset of $\mathbb{R}^d$, and it inherits the relative topology from $\mathbb{R}^d$. To be more explicit, let us consider a sequence $\{\boldsymbol{x}_k : k \in \mathbb{N}\} \subset \mathsf{L}$ and a point $\boldsymbol{x} \in \mathsf{L}$. Then $\boldsymbol{x}_k \to \boldsymbol{x}$ if and only if $\|\boldsymbol{x}_k - \boldsymbol{x}\| \to 0$. (Since the affine space $\mathsf{L}$ is closed, the limit $\boldsymbol{x}$, if it exists, must remain in $\mathsf{L}$.)

### 1.4.2 Affine Hulls

We can generalize the definition (1.4.1) of a line to allow for more than two points. The key is to restrain the sum of the scalar coefficients to equal one.

**Definition 1.4.2** (Affine combination)**.** Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in \mathbb{R}^d$. An *affine combination* of these points takes the form

$$\sum\nolimits_{i=1}^{k} \alpha_i \boldsymbol{x}_i \quad \text{where} \quad \sum\nolimits_{i=1}^{k} \alpha_i = 1 \quad \text{for } \alpha_i \in \mathbb{R}.$$

A short calculation shows that an affine space is a set that contains all of the affine combinations of its points. Let us consider the operation that collects all possible affine combinations of the points in a general set.

**Definition 1.4.3** (Affine hull)**.** Let $\mathsf{A} \subset \mathbb{R}^d$ be a nonempty set. The *affine hull* $\operatorname{aff} \mathsf{A}$ consists of all (finite) affine combinations of points drawn from the set $\mathsf{A}$.

$$\operatorname{aff} \mathsf{A} := \left\{ \sum\nolimits_{i=1}^{k} \alpha_i \boldsymbol{x}_i : \sum\nolimits_{i=1}^{k} \alpha_i = 1 \text{ and } \boldsymbol{x}_i \in \mathsf{A} \text{ and } k \in \mathbb{N} \right\}.$$

It is routine to check that $\operatorname{aff} \mathsf{A}$ is an affine space. See Figure 1.1 for an illustration.

As a simple example, the affine hull of a pair of points is the line spanned by the points: $\operatorname{aff}\{\boldsymbol{x}, \boldsymbol{y}\} = \operatorname{line}(\boldsymbol{x}, \boldsymbol{y})$. It is easy to verify that the affine hull of a set of $k+1$ points has dimension no greater than $k$.

The affine hull, $\operatorname{aff} \mathsf{A}$, of a set $\mathsf{A}$ is the intersection of all affine spaces that contain $\mathsf{A}$. In this sense, it is the smallest affine space containing $\mathsf{A}$.

### 1.4.3 Affine Independence

When we form the affine hull, aff A, of a set A, some of the points may be superfluous. For example, we do not need all the points in a line to form its affine hull: $\operatorname{aff}\operatorname{line}(\boldsymbol{x}, \boldsymbol{y}) = \operatorname{aff}\{\boldsymbol{x}, \boldsymbol{y}\}$. It is valuable to have a notion of when a family of points cannot be reduced without changing its affine hull.

**Definition 1.4.4** (Affine independence)**.** A family $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_k\} \subset \mathbb{R}^d$ is *affinely independent* if none of the points is an affine combination of the others. Algebraically, affine independence is the condition that

$$\sum\nolimits_{i=1}^{k} \alpha_i \boldsymbol{x}_i = 0 \quad \text{and} \quad \sum\nolimits_{i=1}^{k} \alpha_i = 0 \quad \text{imply that} \quad \alpha_i = 0 \text{ for each } i = 1, \dots, k.$$

Affine independence is also equivalent to the family of secants $\{\boldsymbol{x}_1 - \boldsymbol{x}_0, \dots, \boldsymbol{x}_k - \boldsymbol{x}_0\}$ being linearly independent.

It is clear that affine independence is a hereditary property: a subset of an affinely independent set remains affinely independent.

Affinely independent sets generate affine spaces of maximal dimension. Consider a set $\mathsf{A} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_k\} \subset \mathbb{R}^d$. The dimension of the affine hull aff A is no greater than $k$. This bound is achieved precisely when A is affinely independent. In particular, $d + 1$ is the maximum cardinality of an affinely independent set in $\mathbb{R}^d$.

### 1.4.4 Affine Maps

We also need to introduce collection of maps that respect affine geometry. Just as a linear map preserves linear combinations, an affine map preserves affine combinations.

**Definition 1.4.5** (Affine map)**.** A function $\boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^m$ is called an *affine map* if

$$\boldsymbol{T}\Big((1-\alpha)\boldsymbol{x} + \alpha\boldsymbol{y}\Big) = (1-\alpha)\boldsymbol{T}\boldsymbol{x} + \alpha\boldsymbol{T}\boldsymbol{y} \quad \text{when} \quad \alpha \in \mathbb{R} \quad \text{and} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d.$$

This condition is strictly weaker than linearity.

Each affine map $\boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^m$ is the composition of a linear map $\boldsymbol{S} : \mathbb{R}^d \to \mathbb{R}^m$ and translation by a point $\boldsymbol{x}_0 \in \mathbb{R}^d$:

$$\boldsymbol{T}\boldsymbol{x} = \boldsymbol{S}(\boldsymbol{x} - \boldsymbol{x}_0) = \boldsymbol{S}\boldsymbol{x} - \boldsymbol{y}_0 \quad \text{where} \quad \boldsymbol{y}_0 = \boldsymbol{S}\boldsymbol{x}_0 \in \mathbb{R}^m.$$

As a consequence, we can bring to bear the theory of linear maps to understand fully the structure of affine maps.

## 1.5 A First Look at Convex Sets

We begin our journey into the theory of convexity by developing an "internal" view of convex sets. In other words, we study the relationship between a convex set and its own points. Our presentation has strong parallels with the treatment of affine geometry.

### 1.5.1 Line Segments

As we extend our scope beyond affine geometry, the first step is to narrow our attention from lines to line segments.

**Definition 1.5.1** (Line segments). Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ be points. The *closed segment* and *open segment* generated by $\boldsymbol{x}$ and $\boldsymbol{y}$ are, respectively, the sets

$$[\boldsymbol{x}, \boldsymbol{y}] := \{(1-\tau)\boldsymbol{x} + \tau\boldsymbol{y} : \tau \in [0,1]\};$$
$$(\boldsymbol{x}, \boldsymbol{y}) := \{(1-\tau)\boldsymbol{x} + \tau\boldsymbol{y} : \tau \in (0,1)\}.$$

We also define the *half-open segments*:

$$(\boldsymbol{x}, \boldsymbol{y}] := \{(1-\tau)\boldsymbol{x} + \tau\boldsymbol{y} : \tau \in (0,1]\};$$
$$[\boldsymbol{x}, \boldsymbol{y}) := \{(1-\tau)\boldsymbol{x} + \tau\boldsymbol{y} : \tau \in [0,1)\}.$$

As you may surmise, the quantity $1-\tau$ will arise often enough to merit notation:

$$\bar{\tau} := 1 - \tau \quad \text{for } \tau \in \mathbb{R}.$$

Whereas every point in a line is the same, the endpoints of a line segment are distinguished. This apparently innocuous difference explains why convex geometry is a richer subject than affine geometry.

Here is another salient observation. The definition (1.4.1) of a line places no restriction on the value of the parameter $\alpha$. On the other hand, the definition of a line segment restricts the parameter $\tau$ to the interval $[0,1] = \{t \in \mathbb{R} : 0 \leq t \leq 1\}$. We already begin to see that convexity is the realm of linear inequalities, while affine geometry is the realm of linear equalities.

More generally, an affine space is the solution to a system of inhomogeneous linear equations. Later, we will discover that a (closed) convex set is the solution to an (infinite) system of linear inequalities.

### 1.5.2 Convex Sets

We are now prepared to make the fundamental definition of this course.

**Definition 1.5.2** (Convex set). A set $\mathsf{C} \subset \mathbb{R}^d$ is *convex* if it contains all of its line segments:

$$\boldsymbol{x}, \boldsymbol{y} \in \mathsf{C} \quad \text{implies} \quad [\boldsymbol{x}, \boldsymbol{y}] \subset \mathsf{C}.$$

An alternative statement is that, if a convex set contains two points, it also contains all of their arithmetical averages. Note that the empty set $\emptyset$ is (vacuously) convex.

The simplest example of a set that is *not* convex consists of two distinct points. A creative individual will quickly find other sets, such as a kidney bean, that fail to be convex.

Here is a collection of some important convex sets that you may already have encountered. We will permanently affix notation to some of these examples.

1. **Affine spaces.** Each affine space is convex.

2. **Orthant.** The vectors with nonnegative entries form a convex set:
$$\mathbb{R}^d_+ := \{\boldsymbol{x} \in \mathbb{R}^d : x_i \geq 0 \text{ for each } i\}.$$
The vectors with strictly positive entries form a convex set:
$$\mathbb{R}^d_{++} := \{\boldsymbol{x} \in \mathbb{R}^d : x_i > 0 \text{ for each } i\}.$$
These sets are also *convex cones*, as you will see on your homework.
3. **Probability simplex.** The set of probability distributions supported on $\{1, \dots, d\}$ forms a convex set, called the *probability simplex*:
$$\Delta_d := \left\{\boldsymbol{x} \in \mathbb{R}^d_+ : \sum\nolimits_{i=1}^d x_i = 1\right\}.$$
4. **Euclidean ball.** The vectors with norm no greater than one forms a convex set:
$$\mathsf{B}_d := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| \leq 1\}.$$
5. **Norm balls.** More generally, the unit ball of any norm on $\mathbb{R}^d$ is a convex set.
6. **Unit-volume cube.** The unit-volume cube is convex:
$$\mathsf{Q}_d := [0,1]^d = \{\boldsymbol{x} \in \mathbb{R}^d : 0 \leq x_i \leq 1 \text{ for each } i\}.$$
7. **Nonnegative polynomials.** The nonnegative polynomials with fixed maximum degree form a convex cone:
$$\{p \in \mathscr{P}_d : p(t) \geq 0 \text{ for all } t \in \mathbb{R}\}.$$
The linear space $\mathscr{P}_d$ contains all polynomials with real coefficients and degree at most $d$.
8. **Birkhoff polytope.** The doubly stochastic matrices forms a convex set:
$$\left\{\boldsymbol{A} \in \mathbb{R}^{d\times d}_+ : \sum\nolimits_{j=1}^d a_{ij} = 1 \text{ for each } j \text{ and } \sum\nolimits_{i=1}^d a_{ij} \text{ for each } i\right\}.$$
These are nonnegative matrices whose row sums and column sums all equal one.
9. **Positive-semidefinite matrices.** The complex positive-semidefinite (psd) matrices form a convex cone:
$$\mathbb{H}^d_+ := \{\boldsymbol{A} \in \mathbb{C}^{d\times d} : \langle \boldsymbol{x},\, \boldsymbol{A}\boldsymbol{x}\rangle \geq 0 \text{ for all } \boldsymbol{x} \in \mathbb{C}^d\}.$$
In this expression, we have used the standard *Hermitian* inner product. Complex strictly positive-definite (pd) matrices form a convex cone. The real psd matrices also form a closed convex cone, as do the real pd matrices.
10. **Quantum states.** The (complex) psd matrices with trace one form a convex set:
$$\left\{\boldsymbol{A} \in \mathbb{H}^d_+ : \sum\nolimits_{i=1}^d a_{ii} = 1\right\}.$$
This is Richard Kueng's favorite example.

You can see from the diversity of this (very small) collection of examples that convexity arises in a wide range of situations.

### 1.5.3 Operations Preserving Convexity

There are many operations that preserve convexity. Suppose that $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$ are convex. Let $\mathsf{A}, \mathsf{B} \subset \mathbb{R}^d$ and $\mathsf{M} \subset \mathbb{R}^m$ be arbitrary sets.

1. **Intersection.** The set $\mathsf{C} \cap \mathsf{K}$ is convex.
2. **Direct product.** The set $\mathsf{C} \times \mathsf{K}$ is convex.
3. **Dilation.** The dilation $\alpha\mathsf{C}$ is convex for each $\alpha \in \mathbb{R}$. We define the dilation of a general set as
$$\alpha\mathsf{A} := \{\alpha \boldsymbol{x} : \boldsymbol{x} \in \mathsf{A}\}.$$
4. **(Minkowski) addition.** The sum $\mathsf{C} + \mathsf{K}$ is convex. We define the sum of two sets as
$$\mathsf{A} + \mathsf{B} := \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in \mathsf{A} \text{ and } \boldsymbol{y} \in \mathsf{B}\}.$$
This is an unexpectedly rich operation that will repay our investment.
5. **Affine maps.** For any affine map $\boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^m$, the image $\boldsymbol{T}\mathsf{C}$ and the preimage $\boldsymbol{T}^{(-1)}\mathsf{C}$ are convex. Images and preimages are defined as
$$\boldsymbol{T}\mathsf{A} := \{\boldsymbol{T}\boldsymbol{x} : \boldsymbol{x} \in \mathsf{A}\} \quad \text{and} \quad \boldsymbol{T}^{(-1)}\mathsf{M} := \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{T}\boldsymbol{x} \in \mathsf{M}\}.$$

You will verify these claims on your homework.

### 1.5.4 Convex Hulls

The line segment $[\boldsymbol{x}, \boldsymbol{y}]$ is the collection of (arithmetical) averages of the points $\boldsymbol{x}, \boldsymbol{y}$. We can also consider averages of greater numbers of points. This idea leads to a fruitful definition.

**Definition 1.5.3** (Convex combination)**.** Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in \mathbb{R}^d$. A *convex combination* of these points takes the form
$$\sum\nolimits_{i=1}^k \lambda_i \boldsymbol{x}_i \quad \text{where} \quad \sum\nolimits_{i=1}^k \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for each } i.$$
Observe that the vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$ of coefficients belongs to the probability simplex $\Delta_k$.

A set is convex precisely when it contains all convex combinations of points drawn from the set. We may now introduce an operation that collects all possible convex combinations of points from a general set.

**Definition 1.5.4** (Convex hull)**.** Let $\mathsf{A} \subset \mathbb{R}^d$ be a nonempty set. The *convex hull* $\operatorname{conv} \mathsf{A}$ consists of all (finite) convex combinations of points drawn from the set $\mathsf{A}$.
$$\operatorname{conv} \mathsf{A} := \left\{ \sum\nolimits_{i=1}^k \lambda_i \boldsymbol{x}_i : \boldsymbol{\lambda} \in \Delta_k \text{ and } \boldsymbol{x}_i \in \mathsf{A} \text{ and } k \in \mathbb{N} \right\}.$$
It is not hard to check that $\operatorname{conv} \mathsf{A}$ is indeed a convex set. See Figure 1.1 for an illustration.

Here is the simplest example. The convex hull of a pair of points is the line segment generated by the points: $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\} = [\boldsymbol{x}, \boldsymbol{y}]$.

To create a more vivid picture in your mind, you can think of $\operatorname{conv} \mathsf{A}$ as the set obtained by "shrink-wrapping" the set $\mathsf{A}$. The following facts justify this intuition:

- If $\mathsf{C} \subset \mathbb{R}^d$ is convex, then $\operatorname{conv} \mathsf{C} = \mathsf{C}$. In other words, we extract all of the value from the convex hull operation by applying it once.
- The convex hull, $\operatorname{conv} \mathsf{A}$, is the intersection of all convex sets that contain $\mathsf{A}$.

These facts do require proof, which you should take the time to provide.

We close this subsection with the definition of an important class of convex sets.

**Definition 1.5.5** (Simplex)**.** The convex hull of an affinely independent set of points is called a *simplex.*

Examples of simplices include a point, a line segment, a triangle, a tetrahedron, and so forth. Like affinely independent sets, simplices enjoy a hereditary property. Suppose that $\mathsf{A} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\}$ is affinely independent. For every subset $\mathsf{B} \subset \mathsf{A}$, the convex hull $\operatorname{conv} \mathsf{B}$ is also a simplex.

## 1.6 The Carathéodory Theorem

We saw that the affine hull of a set can be generated by a finite subset of its points, with cardinality one plus the dimension of the affine hull. Convex hulls are more complicated objects than affine hulls, and so a finite number of points may not be sufficient to generate the convex hull of an infinite set. For example, consider the unit circle:

$$\mathbb{S}^1 := \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| = 1\}.$$

The convex hull of $\mathbb{S}^1$ is simply the unit disc $\mathsf{B}_2$. It is easy to see that the unit circle has no finite subset whose convex hull exhausts the unit disc.

As a first step toward understanding the structure of convex hulls, let us establish a very important theorem that emphasizes the role of affinely independent points in constructing convex hulls. Later, we will develop a more complete understanding about which points are needed to generate a convex hull.

**Theorem 1.6.1** (Carathéodory)**.** *Suppose that $\mathsf{A} \subset \mathbb{R}^d$ and $\boldsymbol{x} \in \operatorname{conv} \mathsf{A}$. Then $\boldsymbol{x}$ is a convex combination of affinely independent points drawn from $\mathsf{A}$. In particular, $\boldsymbol{x}$ can be expressed as a convex combination of $d+1$ or fewer points in $\mathsf{A}$.*

*Proof.* Since $\boldsymbol{x} \in \operatorname{conv} \mathsf{A}$, we can express it as a convex combination of points drawn from $\mathsf{A}$:

$$\boldsymbol{x} = \sum\nolimits_{i=1}^k \lambda_i \boldsymbol{x}_i \quad \text{where} \quad \sum\nolimits_{i=1}^k \lambda_i = 1 \quad \text{and} \quad \lambda_i > 0 \quad \text{and} \quad \boldsymbol{x}_i \in \mathsf{A} \quad \text{for each } i.$$

We can surely avoid having any coefficients $\lambda_i$ that equal zero. Moreover, among all such representations, we can select one where the number $k$ of summands is minimal.

In service of finding a contradiction, imagine that the family $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\}$ is affinely *dependent.* Then we can find real coefficients $\alpha_1, \dots, \alpha_k$, not all zero, for which

$$\sum\nolimits_{i=1}^k \alpha_i \boldsymbol{x}_i = \mathbf{0} \quad \text{and} \quad \sum\nolimits_{i=1}^k \alpha_i = 0.$$

Select the smallest index $m$ that validates the inequalities

$$\alpha_m > 0 \quad \text{and} \quad \frac{\lambda_m}{\alpha_m} \le \frac{\lambda_i}{\alpha_i} \quad \text{for each } i \text{ where } \alpha_i > 0.$$

This index $m$ must exist because at least one of the $\alpha_i$ is positive.

Combining the first two displays, we can write the distinguished point $\boldsymbol{x} \in \operatorname{conv} \mathsf{A}$ in the form

$$\boldsymbol{x} = \sum\nolimits_{i=1}^{k} \lambda_i \boldsymbol{x}_i - \frac{\lambda_m}{\alpha_m} \sum\nolimits_{i=1}^{k} \alpha_i \boldsymbol{x}_i = \sum\nolimits_{i=1}^{k} \left( \lambda_i - \frac{\lambda_m}{\alpha_m} \alpha_i \right) \boldsymbol{x}_i =: \sum\nolimits_{i=1}^{k} \beta_i \boldsymbol{x}_i.$$

Observe that $\beta_m = 0$. Meanwhile, our choice of $m$ ensures that $\beta_i \geq 0$ for each index $i$. Next, $\sum_{i=1}^{k} \beta_i = 1$ because the $\lambda_i$ sum to one and the $\alpha_i$ sum to zero.

In other words, we have written $\boldsymbol{x}$ as a convex combination of at most $k-1$ points in $\mathsf{A}$. This contradicts the minimality of $k$. We must conclude that the family $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\}$ is affinely independent. □

An alternative statement of Theorem 1.6.1 is that the convex hull of a set is the union of the simplices generated by the set:

$$\operatorname{conv} \mathsf{A} = \bigcup \{\operatorname{conv} \mathsf{T} : \mathsf{T} \subset \mathsf{A} \text{ is affinely independent}\}.$$

Returning to our two-dimensional example, we see that the unit disc $\mathsf{B}_2$ is the union of all triangles with vertices on the unit circle.

As this observation intimates, Theorem 1.6.1 is a powerful tool for representing points in the convex hull of a large set. It produces valuable information about the geometry and topology of convex hulls. It also serves as a building block for more powerful techniques, including the theorems of Minkowski and Dubins on extremal representations.

Beyond that, Carathéodory's theorem stands as the gateway to the field of combinatorial convex geometry. It leads to easy proofs of other important results, including the theorems of Radon and Helly. Perhaps, you will explore these ideas on your homework.

## 1.7 Topology of Convex Sets

In this course, we are interested in how convexity interacts with volume. Therefore, we will focus our attention on compact, convex sets because these sets all have a well-defined volume. As a preliminary, it is valuable to develop a more detailed understanding of the topology of convex sets and convex hulls.

### 1.7.1 The Interior and Closure of a Convex Set

The most basic fact about the topology of a convex set explains how the line segments in a convex set interact with its boundary.

**Proposition 1.7.1** (Internal segments)**.** *Let $\mathsf{C} \subsetneq \mathbb{R}^d$ be a nonempty convex set. Let $\boldsymbol{x} \in \operatorname{int} \mathsf{C}$ and $\boldsymbol{y} \in \operatorname{bd} \mathsf{C}$. Then the open segment generated by these two points lies in the interior of the set: $(\boldsymbol{x}, \boldsymbol{y}) \subset \operatorname{int} \mathsf{C}$.*

The proof is based on an elementary geometric argument, which we leave to you. This simple result has some important consequences:

**Corollary 1.7.2** (Interior and closure of a convex set)**.** *Let $\mathsf{A} \subset \mathbb{R}^d$ be a convex set. Then the interior, $\operatorname{int} \mathsf{A}$, and the closure, $\operatorname{cl} \mathsf{A}$, are both convex sets.*

The convex hull operation also preserves some topological properties.

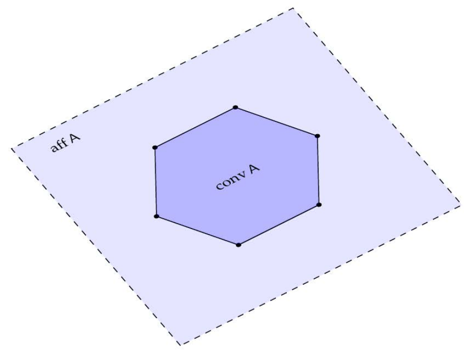


**Figure 1.1** (The affine hull, the convex hull, the relative interior, and the relative boundary). The set $\mathsf{A}$ contains six points (solid circles) that form a hexagon. The affine hull, $\operatorname{aff} \mathsf{A}$, is the plane that contains this hexagon (light blue), The convex hull, $\mathsf{C} := \operatorname{conv} \mathsf{A}$, is the entire hexagon, including its edges (dark blue and black lines). The relative interior, $\operatorname{relint} \mathsf{C}$, is the inside of the hexagon (dark blue), while the relative boundary, $\operatorname{relbd} \mathsf{C}$ consists of the outside of the hexagon (black lines). The set $\mathsf{C}$ has dimension two.

**Theorem 1.7.3** (Topology of a convex hull)**.** *The convex hull of an open set is open. The convex hull of a compact set is compact.*

The first part is an easy exercise, while the second part involves Carathéodory's theorem. We omit the details.

Warning 1.7.4 (Convex hull does not preserve closedness). The convex hull of a closed set need not be closed. For example, consider the convex hull of a line and a point not on the line.

Remark 1.7.5 (Closed convex hull). The closed convex hull of $\mathsf{A}$ equals the intersection of all closed convex sets that contain $\mathsf{A}$.

### 1.7.2 The Dimension of a Convex Set

Convex sets can be flat, like a pancake or a halibut. Indeed, a convex set need not have any interior points. As a consequence, the topological interior and boundary of a convex set are not always the most useful way to think about its inside and outside. Here is a better approach.

**Definition 1.7.6** (Relative interior and boundary)**.** Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty convex set with affine hull $\mathsf{L} = \operatorname{aff} \mathsf{C}$. The *relative interior*, $\operatorname{relint} \mathsf{C}$, is the interior of the set $\mathsf{C}$, computed with respect to the relative topology on $\mathsf{L}$. Since $\mathsf{L}$ is closed in $\mathbb{R}^d$, the closure of $\mathsf{C}$ in $\mathsf{L}$ is the same as its closure in $\mathbb{R}^d$. The *relative boundary*, $\operatorname{relbd} \mathsf{C} := \operatorname{cl} \mathsf{C} \setminus \operatorname{relint} \mathsf{C}$, is the boundary of the set $\mathsf{C}$, computed with respect to the relative topology on $\mathsf{L}$. See Figure 1.1 for an illustration.

A point in the relative interior of a convex set is sometimes called an *internal point* of the set. To appreciate why the relative interior is the correct notion of "inside" for convex sets, we begin with the example of a simplex.

**Proposition 1.7.7** (Relative interior of a simplex). *Consider an affinely independent set* $\mathsf{A} := \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\} \subset \mathbb{R}^d$. *The relative interior of the simplex* $\mathsf{T} = \operatorname{conv} \mathsf{A}$ *takes the form*

$$\operatorname{relint} \mathsf{T} = \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i : \sum_{i=1}^{k} \lambda_i = 1 \text{ and } \lambda_i > 0 \right\}.$$

*In particular, the simplex* $\mathsf{T}$ *has an internal point.*

*Proof sketch.* We may assume that $k = d + 1$. Since $\mathsf{A}$ is affinely independent, the following linear system has a unique solution for each $\boldsymbol{x} \in \operatorname{aff} \mathsf{A}$:

$$\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1.$$

Indeed, the solution operator $\boldsymbol{\lambda} : \operatorname{aff} \mathsf{A} \to \mathbb{R}^k$ is a continuous affine map, called the *barycentric coordinates*. By continuity, if $\boldsymbol{\lambda}(\boldsymbol{x}) > \mathbf{0}$, then $\boldsymbol{\lambda}(\boldsymbol{y}) > \mathbf{0}$ for all $\boldsymbol{y}$ in some open ball $\mathsf{N}(\boldsymbol{x}; r)$. In other words, $\boldsymbol{\lambda}(\boldsymbol{x}) > \mathbf{0}$ implies that $\boldsymbol{x} \in \operatorname{int} \mathsf{T}$.

On the other hand, suppose that $\boldsymbol{x} \in \mathsf{T}$, but $\lambda_i(\boldsymbol{x}) = 0$ for some index $i$. Since $\boldsymbol{\lambda}$ is continuous and affine, each open ball $\mathsf{N}(\boldsymbol{x}; r)$ includes a point $\boldsymbol{y}$ where $\lambda_i(\boldsymbol{y}) < 0$. This point $\boldsymbol{y} \notin \mathsf{T}$. Therefore, $\boldsymbol{x} \in \operatorname{bd} \mathsf{T}$. □

**Corollary 1.7.8** (Relative interior of a convex set). *In* $\mathbb{R}^d$, *each nonempty convex set has an internal point.*

*Proof.* A nonempty convex set coincides with its convex hull. According to Theorem 1.6.1, the convex hull of a set contains a simplex with the same dimension as the affine hull of the set. Proposition 1.7.7 asserts that this simplex contains an internal point. This distinguished internal point must also lie internal to the original convex set. □

To summarize, every nonempty convex set has an “inside” so long as we treat it as a subset of its affine hull. As a consequence, the affine hull of a convex set is its natural milieu. This leads to another important definition.

**Definition 1.7.9** (Dimension of a convex set). In $\mathbb{R}^d$, the *dimension* of a convex set is the dimension of its affine hull.

## 1.8 Extreme Points and Faces

We will invest a lot of energy to comprehend the structure of the boundary of a convex set. Today, we will consider (sets of) points on the boundary of a convex set as they relate to the other points inside the convex set. In the next lecture, we will examine how the boundary of a convex set looks from the outside.

### 1.8.1 Extreme Points

The internal points of a convex set are averages of other points in the convex set. It is interesting to single out points in the convex set that *cannot* be written as averages of other points. We imagine that these are the points in the boundary that are most remote from the interior of the set. In this discussion, we restrict our attention to closed convex sets.

**Definition 1.8.1** (Extreme point). Let $\mathsf{C} \subset \mathbb{R}^d$ be a closed convex set. A point $\boldsymbol{x} \in \mathsf{C}$ is called an *extreme point* of $\mathsf{C}$ if $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C}$ and $\frac{1}{2}(\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x}$ imply that $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{x}$. See Figure 1.2 for an illustration.

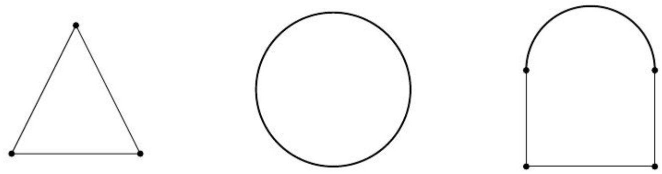

**Figure 1.2** (Faces of the triangle, the unit disc, and the arch). Extreme points are marked with heavy lines and circles, while one-dimensional faces are marked with light lines. [left] The extreme points of the triangle are the three vertices; its one-dimensional faces are the three edges. [center] The extreme points of the unit disc compose the entire unit circle; the unit disc has no one-dimensional faces. [right] The extreme points of the arch are the two lower vertices and the entire top arc; the three one-dimensional faces are the bottom, left, and right sides.

As our intuition suggests, every extreme point belongs to the (relative) boundary of the convex set.

**Proposition 1.8.2** (Extreme points lie in the boundary)**.** *Let* $\mathsf{C} \subset \mathbb{R}^d$ *be a closed convex set. If* $\boldsymbol{x} \in \mathsf{C}$ *is an extreme point, then* $\boldsymbol{x} \in \operatorname{relbd} \mathsf{C}$.

*Proof.* Define $\mathsf{L} = \operatorname{aff} \mathsf{C}$, and suppose that $\boldsymbol{x} \in \operatorname{relint} \mathsf{C}$. By definition of the relative interior, there is a positive number $r$ so that the $\mathsf{L}$-open ball $\mathsf{N}(\boldsymbol{x}; r) \cap \mathsf{L} \subset \mathsf{C}$. In particular, this open ball contains a point $\boldsymbol{x} + \boldsymbol{h}$ that is different from $\boldsymbol{x}$. But then $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{h}$ and $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{h}$ belong to the open ball, and $\boldsymbol{x} = \frac{1}{2}(\boldsymbol{y} + \boldsymbol{z})$. We must conclude that $\boldsymbol{x}$ is not an extreme point of $\mathsf{C}$. □

In the forthcoming lectures, we will come to appreciate the fundamental role of extreme points in determining the structure of a closed, convex set. We will also learn about their importance for optimization.

### 1.8.2 Faces

We can extend the concept of an extreme point by considering a *set* of points in a convex set that cannot be written as the averages of points outside that set.

**Definition 1.8.3** (Face)**.** Let $\mathsf{C} \subset \mathbb{R}^d$ be a closed convex set. A closed convex set $\mathsf{F} \subset \mathsf{C}$ is called a *face* if $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C}$ and $\frac{1}{2}(\boldsymbol{y} + \boldsymbol{z}) \in \mathsf{F}$ implies that $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{F}$. A face is also called an *extreme set*. See Figure 1.2.

**Notation 1.8.4** (Face of ($\lhd$))**.** We sometimes write $\mathsf{F} \lhd \mathsf{C}$ to abbreviate the relation that $\mathsf{F}$ is a face of $\mathsf{C}$.

Since each face is a convex set, we can assign it a dimension. In particular, the extreme points are zero-dimensional faces. By convention, the empty set $\emptyset$ is regarded as a face of a convex set $\mathsf{C}$, and the set $\mathsf{C}$ itself is also a face. The faces $\emptyset, \mathsf{C}$ are called *improper faces* of $\mathsf{C}$; all other faces of $\mathsf{C}$ are called *proper faces*.

As with extreme points, each proper face is contained in the relative boundary of the convex set. The proof is similar.

Faces have very important transitivity and intersection properties. These statements follow quickly from the definition of a face.

**Proposition 1.8.5** (Faces: Transitivity and intersection)**.** *Let* $\mathsf{C} \subset \mathbb{R}^d$ *be a closed convex set.*

1. *If* $\mathsf{F}_1 \lhd \mathsf{C}$ *and* $\mathsf{F}_2 \lhd \mathsf{C}$ *and* $\mathsf{F}_1 \subset \mathsf{F}_2$*, then* $\mathsf{F}_1 \lhd \mathsf{F}_2$.
2. *If* $\mathsf{F}_1 \lhd \mathsf{F}_2$ *and* $\mathsf{F}_2 \lhd \mathsf{C}$*, then* $\mathsf{F}_1 \lhd \mathsf{C}$.
3. *Let* $I$ *be any index set. If* $\mathsf{F}_i \lhd \mathsf{C}$ *for each* $i \in I$*, then the intersection* $\bigcap_{i \in I} \mathsf{F}_i \lhd \mathsf{C}$.

*In particular, the extreme points of a face of a closed convex set are also extreme points of the convex set.*

We will spend quite a lot of time studying how the faces of a convex set interact with each other, and how they relate to its volumetric properties.

Remark 1.8.6 (Faces are closed). It is not necessary to include the qualification that a face is closed. In view of Proposition 1.7.1, the subsequent part of Definition 1.8.3 already implies that each face is closed.

Remark 1.8.7 (Role of $\frac{1}{2}$). In Definitions 1.8.1 and 1.8.3, the number $\frac{1}{2}$ has no special significance. Indeed, a closed convex set $\mathsf{F} \subset \mathsf{C}$ is a face if and only if, for some $\tau \in (0,1)$,

$$\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C} \quad \text{and} \quad \bar{\tau}\boldsymbol{y} + \tau\boldsymbol{z} \in \mathsf{F} \quad \text{implies} \quad \boldsymbol{y}, \boldsymbol{z} \in \mathsf{F}.$$

Warning 1.8.8 (Terminology for faces). The term "face" is not defined consistently in the literature on convex geometry. In particular, some authors use "face" to refer to what we call an "exposed face." Always check the definition!

# Lecture 2: Convex sets from the outside

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*ACM 204, Fall 2018*
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*October 4, 2018*

## 2.1 Agenda for Lecture 2

This lecture takes an external view of convex sets, in contrast to the internal view from Lecture 1. We introduce the Euclidean projector, which reports the closest point in a convex set. The projector is a useful tool for constructing hyperplanes that separate a convex set from an external point. We then use separating hyperplanes to construct supporting hyperplanes, and we learn that every compact, convex set is the intersection of the halfspaces that contain it. We conclude with the concept of an exposed face of a convex set, which is the intersection of the set with a supporting hyperplane.

1. The Euclidean projector
2. Hyperplanes and halfspaces
3. Separating hyperplanes
4. Supporting hyperplanes and the support function
5. Exposed faces

## 2.2 The Euclidean Projector

To begin, we introduce an important function, called the Euclidean projector, that finds the closest point in a closed convex set. This function is a useful tool for studying separation and support; it will also provide an important ingredient for our investigation of volumetric properties.

**Definition 2.2.1** (Distance and projection)**.** Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty closed convex set, and let $\boldsymbol{x} \in \mathbb{R}^d$ be a point. The *distance* from $\boldsymbol{x}$ to $\mathsf{C}$ is

$$\operatorname{dist}(\boldsymbol{x}; \mathsf{C}) := \operatorname{dist}_{\mathsf{C}}(\boldsymbol{x}) := \min\{\|\boldsymbol{y} - \boldsymbol{x}\| : \boldsymbol{y} \in \mathsf{C}\}.$$

A compactness argument shows that the minimum is achieved, and a strict convexity argument shows that the minimizer is unique. Therefore, we may define a function

$$\operatorname{proj}(\boldsymbol{x}; \mathsf{C}) := \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x}) := \arg\min\{\|\boldsymbol{y} - \boldsymbol{x}\| : \boldsymbol{y} \in \mathsf{C}\}.$$

The function $\operatorname{proj}_{\mathsf{C}}(\cdot)$ is called the *projector* onto $\mathsf{C}$, while the value $\operatorname{proj}_{\mathsf{C}}(\boldsymbol{x})$ is called the *projection* of $\boldsymbol{x}$ onto $\mathsf{C}$. The projector is also called the *Euclidean projector* or the *metric projector*.

The projection has an elegant variational characterization that allows us to establish its key properties.

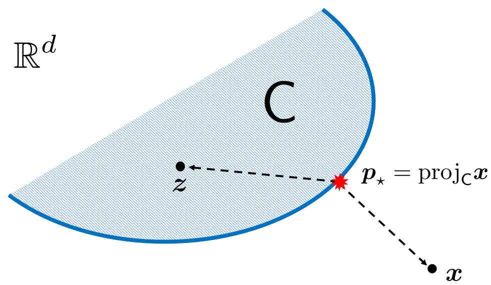


**Figure 2.1** (Euclidean projection of $x$ onto C). The vector connecting the projection $\text{proj}_\mathsf{C}(x)$ to $x$ forms an obtuse angle with every vector connecting $\text{proj}_\mathsf{C}(x)$ to a point $z$ in the set C.

**Theorem 2.2.2** (Variational characterization of projection). *Let* $\mathsf{C} \subset \mathbb{R}^d$ *be a nonempty closed convex set, and fix a point* $p_\star \in \mathsf{C}$. *Then*

$$p_\star = \text{proj}_\mathsf{C}(x) \quad \textit{if and only if} \quad \langle x - p_\star,\, z - p_\star \rangle \le 0 \quad \textit{for all } z \in \mathsf{C}.$$

*See Figure 2.1 for an illustration.*

This result is simply the first-order optimality condition for minimizing a smooth function (i.e., the *squared* distance) over the convex set C. This fact is important for us, so we include a complete proof.

*Proof.* To prove the forward implication, assume that $p_\star = \text{proj}_\mathsf{C}(x) \in \mathsf{C}$. Fix a point $z \in \mathsf{C}$ and a number $\tau \in (0,1)$. By convexity,

$$\bar{\tau} p_\star + \tau z = p_\star + \tau(z - p_\star) \in \mathsf{C}.$$

Since $p_\star$ is the point in C at minimum distance from $x$,

$$\|p_\star - x\| \le \|p_\star + \tau(z - p_\star) - x\|.$$

Square both sides, expand the squared norm on the right-hand side, and rearrange to arrive at

$$2\tau \langle x - p_\star,\, z - p_\star \rangle \le \tau^2 \|z - p_\star\|^2.$$

Divide by $2\tau$ and take the limit as $\tau \to 0$ to discover that

$$\langle x - p_\star,\, z - p_\star \rangle \le 0.$$

Since $z \in \mathsf{C}$ is arbitrary, the forward implication holds.

To prove the reverse implication, assume that $p_\star \in \mathsf{C}$ satisfies the variational condition for each $z \in \mathsf{C}$. We need to show that $p_\star$ is the projection of $x$ onto C. Calculate that

$$\begin{aligned} 0 &\ge \langle x - p_\star,\, (z - x) + (x - p_\star) \rangle \\ &= \|x - p_\star\|^2 + \langle x - p_\star,\, z - x \rangle \\ &\ge \|x - p_\star\|^2 - \|x - p_\star\| \|z - x\|. \end{aligned}$$

The last inequality is Cauchy–Schwarz. Rearranging, we conclude that

$$\|\boldsymbol{p}_\star - \boldsymbol{x}\| \le \|\boldsymbol{z} - \boldsymbol{x}\|.$$

Since $\boldsymbol{z} \in \mathsf{C}$ is arbitrary, we determine that $\boldsymbol{p}_\star$ is the projection of $\boldsymbol{x}$ onto $\mathsf{C}$. □

The variational characterization yields a number of significant consequences. First, we investigate how projections of two different points interact.

**Corollary 2.2.3** (Projection of two points)**.** *Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty closed convex set. For all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,*

$$\|\operatorname{proj}_{\mathsf{C}}(\boldsymbol{x}) - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y})\|^2 \le \langle \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x}) - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y}),\ \boldsymbol{x} - \boldsymbol{y}\rangle.$$

*Proof.* We instantiate the variational characterization from Theorem 2.2.2 twice. For the projection of $\boldsymbol{x}$ onto $\mathsf{C}$, we select the mobile point $\boldsymbol{z} = \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y})$. For the projection of $\boldsymbol{y}$ onto $\mathsf{C}$, we select the mobile point $\boldsymbol{z} = \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x})$. Thus,

$$\langle \boldsymbol{x} - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x}),\ \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y}) - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x})\rangle \le 0;$$
$$\langle \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y}) - \boldsymbol{y},\ \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y}) - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x})\rangle \le 0.$$

We have rewritten the second relation by negating both sides of the inner product. Add the two relations:

$$\langle (\boldsymbol{x} - \boldsymbol{y}) - (\operatorname{proj}_{\mathsf{C}}(\boldsymbol{x}) - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y})),\ \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y}) - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x})\rangle \le 0.$$

Expand the inner product and rearrange to achieve the stated result. □

We may draw two further corollaries. The first result shows that the projection varies in a regular way. It is an immediate consequence of Corollary 2.2.3 and the Cauchy–Schwarz inequality.

**Corollary 2.2.4** (The projector is Lipschitz)**.** *The projector $\operatorname{proj}_{\mathsf{C}}(\cdot)$ onto a closed convex set $\mathsf{C} \subset \mathbb{R}^d$ is a 1-Lipschitz function:*

$$\|\operatorname{proj}_{\mathsf{C}}(\boldsymbol{x}) - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y})\| \le \|\boldsymbol{x} - \boldsymbol{y}\| \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d.$$

*In particular, the projector is continuous.*

Later, we will see that the projection of a point onto a closed convex set $\mathsf{C}$ varies continuously as we change $\mathsf{C}$, but we will have to introduce a topology on convex sets before we can make sense of this statement.

The next fact is tangential to our purposes, but it plays a foundational role in optimization theory; see Rockafellar [Roc70, Sec. 24].

**Corollary 2.2.5** (The projector is monotone)**.** *The projector onto a closed convex set $\mathsf{C} \subset \mathbb{R}^d$ is a monotone operator. That is, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{C}$,*

$$0 \le \langle \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x}) - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y}),\ \boldsymbol{x} - \boldsymbol{y}\rangle.$$

*The map $\mathbf{I} - \operatorname{proj}_{\mathsf{C}}$ is also a monotone operator:*

$$0 \le \langle (\boldsymbol{x} - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{x})) - (\boldsymbol{y} - \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y})),\ \boldsymbol{x} - \boldsymbol{y}\rangle.$$

*(In fact, both maps are maximal monotone.)*

This result follows quickly from Corollaries 2.2.3 and 2.2.4.

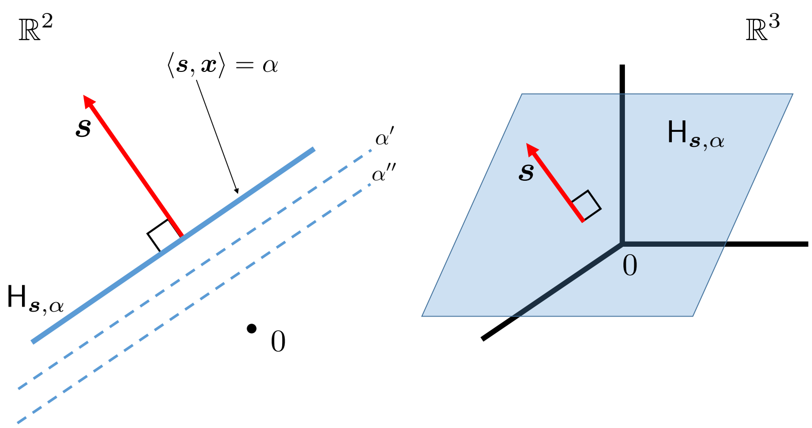


**Figure 2.2** (Hyperplanes). Hyperplanes $\mathsf{H}_{\boldsymbol{s},\alpha}$ and their corresponding normal vectors $\boldsymbol{s}$ and levels $\alpha$. Increasing the level $\alpha$ moves the hyperplane in the direction $\boldsymbol{s}$.

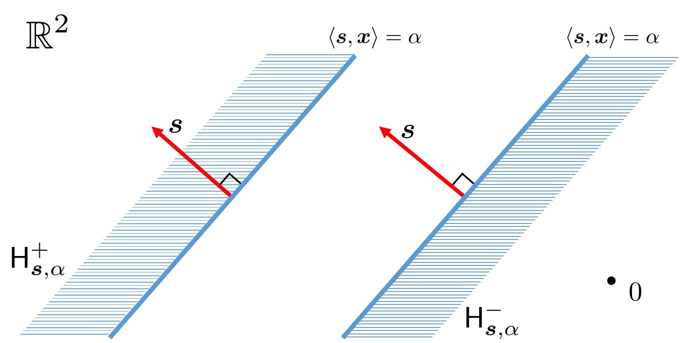


**Figure 2.3** (Halfspaces). [left] The positive halfspace $\mathsf{H}^+_{\boldsymbol{s},\alpha}$ with inner normal $\boldsymbol{s}$ at level $\alpha$. [right] The negative halfspace $\mathsf{H}^-_{\boldsymbol{s},\alpha}$ with outer normal $\boldsymbol{s}$ at level $\alpha$.

## 2.3 Hyperplanes and Halfspaces

Hyperplanes and halfspaces are geometric objects that loom large in the theory of convexity. They give us the language for developing separation and support theorems, which will emerge later in this lecture. First, we need some terminology and notation.

**Definition 2.3.1** (Hyperplane and halfspace)**.** In $\mathbb{R}^d$, a *hyperplane* is an affine space with codimension one. A hyperplane cuts the linear space into two disjoint regions. We define a *closed halfspace* to be the set of all points on one side of a hyperplane, including the points in the hyperplane.

**Notation 2.3.2** (Hyperplane and halfspace)**.** Fix a nonzero vector $\boldsymbol{s} \in \mathbb{R}^d$, called a *normal*, and a scalar $\alpha \in \mathbb{R}$, called a *level*. The hyperplane with normal $\boldsymbol{s}$ at level $\alpha$ is the affine space

$$\mathsf{H}_{\boldsymbol{s},\alpha} := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s},\, \boldsymbol{x}\rangle = \alpha\}.$$

See Figure 2.2 for a picture of some hyperplanes. The (closed) halfspace with *outer* normal $\boldsymbol{s}$ at level $\alpha$ is the closed convex set

$$\mathsf{H}^-_{\boldsymbol{s},\alpha} := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s},\, \boldsymbol{x}\rangle \le \alpha\}.$$

The (closed) halfspace with *inner* normal $\boldsymbol{s}$ at level $\alpha$ is the closed convex set

$$\mathsf{H}^+_{\boldsymbol{s},\alpha} := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s},\, \boldsymbol{x}\rangle \ge \alpha\}.$$

See Figure 2.3 for a diagram of these halfspaces.

We have chosen the letter $\boldsymbol{s}$ as a mnemonic for “slope.” The normal vector $\boldsymbol{s}$ is orthogonal to all vectors contained in its hyperplane:

$$\langle \boldsymbol{s},\, \boldsymbol{x} - \boldsymbol{y}\rangle = 0 \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathsf{H}_{\boldsymbol{s},\alpha}.$$

All hyperplanes with normal $\boldsymbol{s}$ are parallel with each other; increasing the level parameter $\alpha$ translates the hyperplane in the direction $\boldsymbol{s}$. Note that a hyperplane contains the origin if and only if the level $\alpha = 0$.

There is also a valuable algebraic perspective on hyperplanes and halfspaces. Recall that a *linear functional* on $\mathbb{R}^d$ is a linear map $\varphi : \mathbb{R}^d \to \mathbb{R}$. By the representation theorem, the linear functional takes the form $\varphi(\boldsymbol{x}) = \langle \boldsymbol{s},\, \boldsymbol{x}\rangle$ for some vector $\boldsymbol{s} \in \mathbb{R}^d$. As a consequence, we can just as well parameterize hyperplanes and halfspaces by means of (nonzero) linear functionals:

$$\begin{aligned}
\mathsf{H}_{\varphi,\alpha} &:= \{\boldsymbol{x} \in \mathbb{R}^d : \varphi(\boldsymbol{x}) = \alpha\};\\
\mathsf{H}^-_{\varphi,\alpha} &:= \{\boldsymbol{x} \in \mathbb{R}^d : \varphi(\boldsymbol{x}) \le \alpha\};\\
\mathsf{H}^+_{\varphi,\alpha} &:= \{\boldsymbol{x} \in \mathbb{R}^d : \varphi(\boldsymbol{x}) \ge \alpha\}.
\end{aligned}$$

We sometimes adopt the functional perspective in our discussion when it leads to clearer explanations. In the infinite-dimensional setting, linear functionals are indispensable.

## 2.4 Separating Hyperplanes

We continue with an intuitive geometric fact about. Given any point not contained in a closed convex set, we can insert a hyperplane between the point and the set. In a word, we can *separate* a convex set from an external point. This simple idea turns out to be one of the most powerful consequences of convexity. Let us state and prove the basic separation theorem for closed convex sets.

**Theorem 2.4.1** (Proper separation of a point from a closed convex set)**.** *Let $\mathsf{C} \subset \mathbb{R}^d$ be a closed convex set. Fix a point $\boldsymbol{x} \notin \mathsf{C}$. Then there exists a (nonzero) vector $\boldsymbol{s} \in \mathbb{R}^d$ for which*

$$\langle \boldsymbol{s},\, \boldsymbol{x}\rangle > \sup\{\langle \boldsymbol{s},\, \boldsymbol{z}\rangle : \boldsymbol{z} \in \mathsf{C}\}.$$

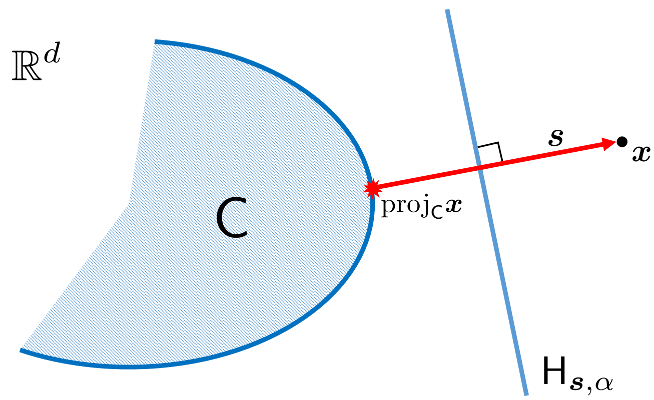


**Figure 2.4** (Separating hyperplane). The hyperplane $\mathsf{H}_{\boldsymbol{s},\alpha}$ properly separates the closed convex set $\mathsf{C}$ from the external point $\boldsymbol{x}$. The hyperplane is normal to the vector $\boldsymbol{s} = \boldsymbol{x} - \text{proj}_{\mathsf{C}}(\boldsymbol{x})$ emanating from the projection $\text{proj}_{\mathsf{C}}(\boldsymbol{x})$ and connecting to the point $\boldsymbol{x}$.

*Proof.* The Euclidean projector provides a geometrically natural approach to the separation theorem. Introduce the nonzero vector

$$\boldsymbol{s} := \boldsymbol{x} - \text{proj}_{\mathsf{C}}(\boldsymbol{x}).$$

(The vector $\boldsymbol{s}$ is nonzero because $\boldsymbol{x}$ does not belong to the closed set $\mathsf{C}$.) The variational characterization of the projection, Theorem 2.2.2, states that

$$0 \geq \langle \boldsymbol{x} - \text{proj}_{\mathsf{C}}(\boldsymbol{x}),\, \boldsymbol{z} - \text{proj}_{\mathsf{C}}(\boldsymbol{x}) \rangle = \langle \boldsymbol{s},\, \boldsymbol{z} - \boldsymbol{x} + \boldsymbol{s} \rangle \quad \text{for all } \boldsymbol{z} \in \mathsf{C}.$$

Rearranging, we arrive at the bound

$$\langle \boldsymbol{s},\, \boldsymbol{x} \rangle \geq \langle \boldsymbol{s},\, \boldsymbol{z} \rangle + \|\boldsymbol{s}\|^2 \quad \text{for all } \boldsymbol{z} \in \mathsf{C}.$$

Take the supremum over $\boldsymbol{z} \in \mathsf{C}$ to conclude

$$\langle \boldsymbol{s},\, \boldsymbol{x} \rangle \geq \sup\{\langle \boldsymbol{s},\, \boldsymbol{z} \rangle : \boldsymbol{z} \in \mathsf{C}\} + \|\boldsymbol{s}\|^2.$$

Since $\boldsymbol{s}$ is nonzero, the inequality in the statement is strict. □

Let us reinterpret Theorem 2.4.1 geometrically. Choose the level

$$\alpha = \frac{1}{2}\Big[\langle \boldsymbol{s},\, \boldsymbol{x} \rangle + \sup\{\langle \boldsymbol{s},\, \boldsymbol{z} \rangle : \boldsymbol{z} \in \mathsf{C}\}\Big].$$

As shown in Figure 2.4, the hyperplane $\mathsf{H}_{\boldsymbol{s},\alpha}$ lies *strictly* between the point $\boldsymbol{x}$ and the set $\mathsf{C}$, intersecting neither. This picture explains the term "proper separation."

It is often more convenient to describe separation in terms of linear functionals. Using this language, we can make the notion of separation rigorous.

**Definition 2.4.2** (Separation)**.** Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a *nonzero* linear functional. Let $\mathsf{A}, \mathsf{B} \subset \mathbb{R}^d$ be nonempty sets. We say that the linear functional *separates* (for emphasis, *weakly separates*) the sets if

$$\begin{aligned} &\text{either} && \inf\{\varphi(\boldsymbol{x}) : \boldsymbol{x} \in \mathsf{A}\} \geq \sup\{\varphi(\boldsymbol{y}) : \boldsymbol{y} \in \mathsf{B}\} \\ &\text{or} && \inf\{\varphi(\boldsymbol{y}) : \boldsymbol{y} \in \mathsf{B}\} \geq \sup\{\varphi(\boldsymbol{x}) : \boldsymbol{x} \in \mathsf{A}\}. \end{aligned}$$

We say that the linear functional *properly separates* the sets if one of these two conditions holds with a strict inequality. Since hyperplanes are equivalent to linear functionals, we can transfer the terminology of separation to hyperplanes.

Geometrically, separation means that we can insert a hyperplane between the sets, but the hyperplane may contact the boundary of one or both of the sets. Proper separation means that we can insert a hyperplane between the sets, avoiding the boundary of each; this situation obtains in Figure 2.4.

Theorem 2.4.1 states that we can properly separate a point from a closed convex set. Here is a useful extension to two sets.

**Corollary 2.4.3** (Proper separation of convex sets)**.** *Let $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$ be closed convex sets, one of which is compact. If $\mathsf{C} \cap \mathsf{K} = \emptyset$, then we can properly separate the two sets with a hyperplane.*

*Proof sketch.* The difference set $\mathsf{D} := \mathsf{C} - \mathsf{K}$ is closed and convex, and it does not contain the origin. Theorem 2.4.1 yields a linear functional $\varphi$ that properly separates $\mathsf{D}$ from the origin. The same linear functional properly separates $\mathsf{C}$ from $\mathsf{K}$. □

Proper separation requires topological assumptions. But (weak) separation is possible when we only enjoy the benefit of convexity.

**Theorem 2.4.4** (Weak separation of convex sets)**.** *Let $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$ be convex sets whose relative interiors do not intersect: $\operatorname{relint} \mathsf{C} \cap \operatorname{relint} \mathsf{K} = \emptyset$. Then we can (weakly) separate the two sets with a hyperplane.*

We probably will not need this type of separation theorem, so we omit the proof.

## 2.5 Supporting Hyperplanes and Halfspaces

We can also use hyperplanes to help understand the boundary structure of a convex set. To do so, we will consider the hyperplanes that (weakly) separate a point on the boundary from the set.

**Definition 2.5.1** (Supporting hyperplane)**.** Let $\mathsf{A} \subset \mathbb{R}^d$ be a set, and let $\boldsymbol{x} \in \operatorname{bd} \mathsf{A}$. A hyperplane $\mathsf{H}_{\boldsymbol{s},\alpha}$ *supports* the set $\mathsf{A}$ at the boundary point $\boldsymbol{x}$ if the hyperplane contains the point $\boldsymbol{x}$ and the set $\mathsf{A}$ is contained in the negative halfspace:

$$\boldsymbol{x} \in \mathsf{H}_{\boldsymbol{s},\alpha} \quad \text{and} \quad \mathsf{A} \subset \mathsf{H}^{-}_{\boldsymbol{s},\alpha}.$$

By convention, we require the normal $\boldsymbol{s}$ to point outward from the set $\mathsf{A}$. Equivalently, a nonzero linear functional $\varphi : \mathbb{R}^d \to \mathbb{R}$ supports $\mathsf{A}$ at $\boldsymbol{x}$ if

$$\varphi(\boldsymbol{x}) = \sup\{\varphi(\boldsymbol{z}) : \boldsymbol{z} \in \mathsf{A}\}.$$

See Figure 2.5 for a diagram. See also Remark 2.5.6 below.

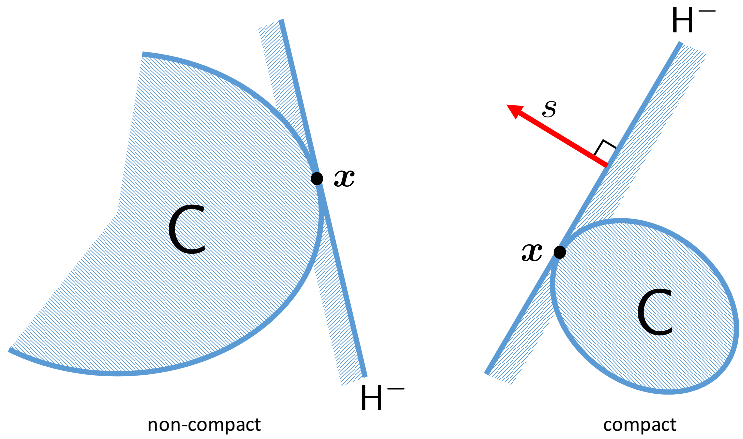


**Figure 2.5** (Supporting halfspaces). [left] The halfspace $\mathsf{H}^-$ supports the convex set $\mathsf{C}$ at the point $\boldsymbol{x}$. [right] If $\mathsf{C}$ is compact and convex and $\boldsymbol{s} \neq \mathbf{0}$, then there is a supporting halfspace $\mathsf{H}^-$ with outer normal $\boldsymbol{s}$.

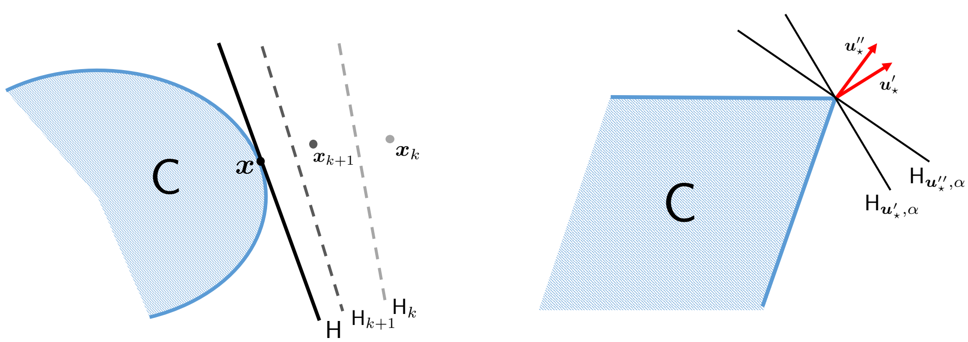


**Figure 2.6** (Separating hyperplanes converging to a supporting hyperplane). [left] The sequence of external points $\boldsymbol{x}_k \notin \mathsf{C}$ converges to a boundary point $\boldsymbol{x}$ of the closed convex set $\mathsf{C}$. The sequence $\mathsf{H}_k$ of hyperplanes separating $\boldsymbol{x}_k$ from $\mathsf{C}$ converges to a hyperplane $\mathsf{H}$ that supports $\mathsf{C}$ at $\boldsymbol{x}$. [right] A convex set $\mathsf{C}$ need not have a unique supporting hyperplane at a given boundary point.

The next result demonstrates that a proper convex set is supported at every point of its boundary. This fact complements Theorem 2.5.3.

**Theorem 2.5.2** (Supporting hyperplanes to a convex set). *Let $\mathsf{C} \subsetneq \mathbb{R}^d$ be a nonempty convex set. For each $\boldsymbol{x} \in \operatorname{bd} \mathsf{C}$, there is a hyperplane that supports $\mathsf{C}$ at $\boldsymbol{x}$.*

*Proof.* The key idea is to produce the supporting hyperplane as a limit of separating hyperplanes. This effort requires some care because the limit may not exist when the boundary is nonsmooth. See Figure 2.6.

Let $\boldsymbol{x} \in \operatorname{bd} \mathsf{C}$. By definition of the boundary, for each $r > 0$, we can describe an open ball $\mathsf{N}(\boldsymbol{x}, r)$ about $\boldsymbol{x}$ that exits $\operatorname{cl} \mathsf{C}$. As a consequence, we can extract a sequence from the exterior of the set $\operatorname{cl} \mathsf{C}$ that converges to the distinguished boundary point $\boldsymbol{x}$:

$$\boldsymbol{x}_k \to \boldsymbol{x} \quad \text{and} \quad \boldsymbol{x}_k \in \mathbb{R}^d \setminus \operatorname{cl} \mathsf{C} \quad \text{for each } k \in \mathbb{N}.$$

For each index $k$, Theorem 2.4.1 yields a hyperplane with normal $\boldsymbol{s}_k \neq \mathbf{0}$ that separates $\boldsymbol{x}_k$ from $\operatorname{cl} \mathsf{C}$, hence from $\mathsf{C}$ (because the set is contained in its closure).

Rescale each normal $\boldsymbol{u}_k := \boldsymbol{s}_k / \|\boldsymbol{s}_k\|$ to obtain a unit vector. Since the unit sphere in $\mathbb{R}^d$ is compact, we can extract a convergent subsequence from the sequence of unit normal vectors: $\boldsymbol{u}_{k_\ell} \to \boldsymbol{u}_\star$ where $\|\boldsymbol{u}_\star\| = 1$.

For each index $k_\ell$, the unit normal vector $\boldsymbol{u}_{k_\ell}$ also generates a hyperplane that separates $\boldsymbol{x}_{k_\ell}$ from $\mathsf{C}$:

$$\langle \boldsymbol{u}_{k_\ell}, \boldsymbol{x}_{k_\ell} - \boldsymbol{z} \rangle > 0 \quad \text{for each } \boldsymbol{z} \in \mathsf{C}.$$

Take the limit as the subindex $\ell \to \infty$ to see that

$$\langle \boldsymbol{u}_\star, \boldsymbol{x} - \boldsymbol{z} \rangle \geq 0 \quad \text{for each } \boldsymbol{z} \in \mathsf{C}.$$

In other words,

$$\alpha := \langle \boldsymbol{u}_\star, \boldsymbol{x} \rangle \geq \sup\{\langle \boldsymbol{u}_\star, \boldsymbol{z} \rangle : \boldsymbol{z} \in \mathsf{C}\}.$$

We conclude that the hyperplane $\mathsf{H}_{\boldsymbol{u}_\star, \alpha}$ supports $\mathsf{C}$ at $\boldsymbol{x}$. □

For a compact convex set, it is easy to see that every possible normal vector generates a supporting hyperplane. Boundedness is essential for this type of result because an unbounded convex set is not supported in its directions of recession.

**Theorem 2.5.3** (Supporting hyperplanes to a compact convex set)**.** *Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty compact convex set. For each $\boldsymbol{s} \neq \mathbf{0}$, there is a supporting hyperplane to $\mathsf{C}$ with normal $\boldsymbol{s}$. See Figure 2.5.*

*Proof.* Fix a direction $\boldsymbol{s} \in \mathbb{R}^d$. Since $\mathsf{C}$ is compact, we can define the level

$$\alpha_\star := \max\{\langle \boldsymbol{s}, \boldsymbol{z} \rangle : \boldsymbol{z} \in \mathsf{C}\}.$$

The maximum is achieved at some point $\boldsymbol{x}_\star \in \mathsf{C}$. We quickly verify that the hyperplane $\mathsf{H}_{\boldsymbol{s}, \alpha_\star}$ supports the set $\mathsf{C}$ at the point $\boldsymbol{x}_\star$. □

Theorem 2.5.3 has a remarkable consequence. It provides an external description of a compact convex set as the intersection of all halfspaces that contain the set.

**Corollary 2.5.4** (Dual representation of a compact convex set)**.** *Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty compact convex set. Then*

$$\mathsf{C} = \bigcap \{\mathsf{H}_- : \mathsf{H}^- \text{ is a halfspace containing } \mathsf{C}\}.$$

*A fortiori,*

$$\mathsf{C} = \bigcap \{\mathsf{H}_- : \mathsf{H}^- \text{ is a supporting halfspace of } \mathsf{C}\}.$$

*See Figure 2.7 for a depiction.*

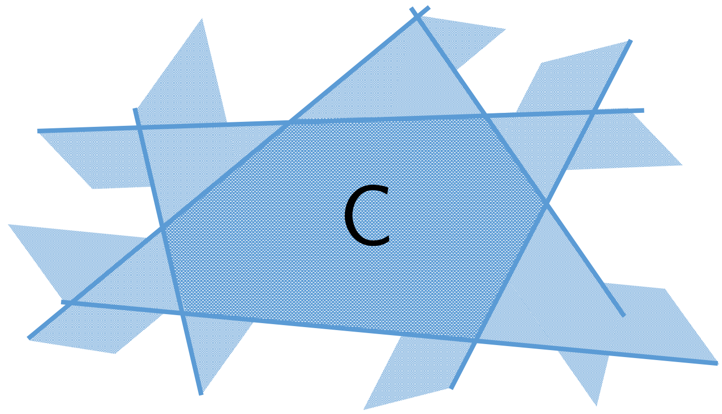


**Figure 2.7** (Dual representation of a convex set). A compact convex set is the intersection of the closed halfspaces that contain it.

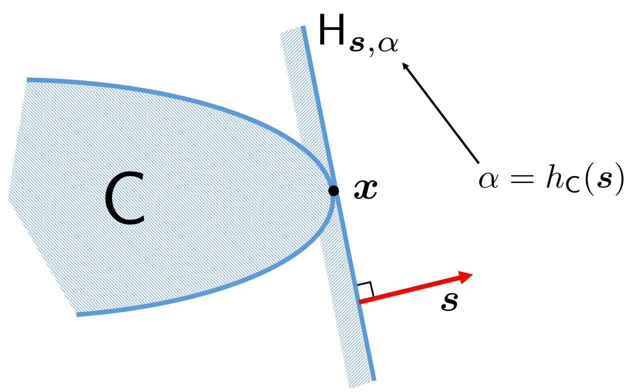


**Figure 2.8** (Support function). Given a nonzero normal vector $\boldsymbol{s} \in \mathbb{R}^d$, we compute the level $\alpha = h_{\mathsf{C}}(\boldsymbol{s})$. If $\alpha < +\infty$, the hyperplane $\mathsf{H}_{\boldsymbol{s},\alpha}$ supports the convex set $\mathsf{C}$.

*Proof sketch.* Suppose that the intersection of supporting halfspaces contains a point $\boldsymbol{z}$ that is external to $\mathsf{C}$. Use Theorem 2.4.1 to separate $\boldsymbol{z}$ from $\mathsf{C}$ by a hyperplane $\mathsf{H}_\star$. By Theorem 2.5.3, the set $\mathsf{C}$ has a supporting hyperplane parallel with $\mathsf{H}_\star$. But then one of the halfspaces of $\mathsf{H}_\star$ contains $\mathsf{C}$ and excludes $\boldsymbol{z}$. Therefore, $\boldsymbol{z}$ cannot lie in the intersection after all. □

In fact, every closed convex set is the intersection of its supporting halfspaces. This result takes a bit more bookkeeping, so we omit the proof.

It is convenient to pack up the information about the level of each supporting hyperplane into a function.

**Definition 2.5.5** (Support function)**.** Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty convex set. The *support function*

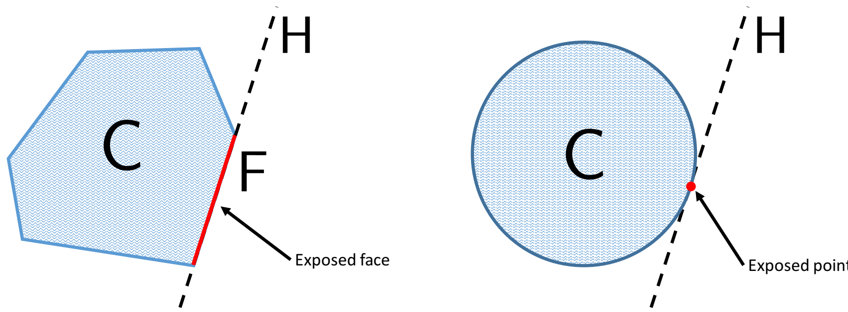


**Figure 2.9** (Exposed faces). [left] The intersection of a closed convex set $\mathsf{C}$ with a supporting hyperplane $\mathsf{H}$ yields an exposed face $\mathsf{F} = \mathsf{C} \cap \mathsf{H}$. [right] An exposed point is a 0-dimensional exposed face.

of the set is defined as

$$h(\boldsymbol{s}; \mathsf{C}) := h_{\mathsf{C}}(\boldsymbol{s}) := \sup\{\langle \boldsymbol{s}, \boldsymbol{z} \rangle : \boldsymbol{z} \in \mathsf{C}\} \quad \text{for } \boldsymbol{s} \in \mathbb{R}^d.$$

In particular, for a unit vector $\boldsymbol{u} \in \mathbb{S}^{d-1}$, the value of the support function $h_{\mathsf{C}}(\boldsymbol{u})$ yields the signed distance from the origin to the supporting hyperplane of $\mathsf{C}$ with normal $\boldsymbol{u}$. See Figure 2.8 for an illustration.

The support function can take values in the extended real numbers. Indeed, if $\mathsf{C}$ has no supporting hyperplane with outer normal $\boldsymbol{s}$, then $h_{\mathsf{C}}(\boldsymbol{s}) = +\infty$. On the other hand, if $\mathsf{C}$ is compact, then the support function is everywhere finite. The support function is a very useful tool, and we will return to investigate its properties in more detail.

Remark 2.5.6 (Proper support). Consider a closed, convex set $\mathsf{C} \subset \mathbb{R}^d$ with $\dim \mathsf{C} < d$. Each hyperplane that contains $\operatorname{aff} \mathsf{C}$ is a supporting hyperplane to $\mathsf{C}$ at every point $\boldsymbol{x}$. This is not so useful. By restricting our attention to the affine hull, we can easily produce supporting hyperplanes that do not contain the entire set; these are called *proper supporting hyperplanes*. Similarly, a linear functional $\varphi$ that is not constant on $\mathsf{C}$ is called a *proper supporting functional* for $\mathsf{C}$.

## 2.6 Exposed Faces

Supporting hyperplanes give us another way to examine the boundary of a convex set. In particular, they lead to another useful notion of "facial structure."

**Definition 2.6.1** (Exposed face). Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty closed convex set, and let $\mathsf{H}$ be a hyperplane that supports $\mathsf{C}$. The closed convex set $\mathsf{F} := \mathsf{C} \cap \mathsf{H}$ is called an *exposed face* of the set $\mathsf{C}$. See Figure 2.9.

We can also define an exposed face of a closed convex set $\mathsf{C}$ as a subset on which a linear functional achieves its maximum value. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a nonzero linear functional. Then a set $\mathsf{F} \subset \mathsf{C}$ is an exposed face if and only if

$$\mathsf{F} = \arg\max\{\varphi(\boldsymbol{x}) : \boldsymbol{x} \in \mathsf{C}\}.$$

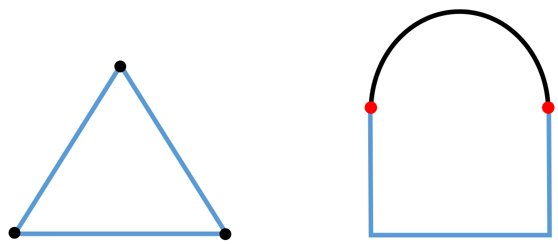

**Figure 2.10** (Exposed points and faces). [left] The vertices of a simplex are exposed points; the edges are exposed faces of dimension one. [right] The red points are extreme points of the arch that are *not* exposed points; the top arc consists of exposed points; the bottom, left, and right sides are exposed faces of dimension one.

This connection indicates why exposed faces play an important role in optimization.

Since an exposed face is a convex set, we can assign it a dimension. In particular, a 0-dimensional exposed face is called an *exposed point*. See Figure 2.10 for an illustration.

The construction of an exposed face via a supporting hyperplane ensures that every exposed face is a subset of the boundary. The terminology "exposed face" is justified by the following result.

**Proposition 2.6.2** (Exposed faces are faces). *Let* $\mathsf{C} \subset \mathbb{R}^d$ *be a closed convex set. If* $\mathsf{F}$ *is an exposed face of* $\mathsf{C}$*, then* $\mathsf{F}$ *is a face of* $\mathsf{C}$*.*

*Proof.* Suppose that $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a nonzero linear functional that generates the exposed face $\mathsf{F}$, and introduce notation for the value $\alpha$ that $\varphi$ achieves on $\mathsf{F}$. That is,

$$\mathsf{F} = \arg\max\{\varphi(\boldsymbol{x}) : \boldsymbol{x} \in \mathsf{C}\} \quad \text{and} \quad \alpha := \max\{\varphi(\boldsymbol{x}) : \boldsymbol{x} \in \mathsf{C}\}.$$

Let us prove that $\mathsf{F}$ is a face. Extract points $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C}$ for which $\frac{1}{2}(\boldsymbol{y} + \boldsymbol{z}) \in \mathsf{F}$. Then

$$\alpha = \varphi\big(\tfrac{1}{2}(\boldsymbol{y} + \boldsymbol{z})\big) = \tfrac{1}{2}\varphi(\boldsymbol{y}) + \tfrac{1}{2}\varphi(\boldsymbol{z}) \leq \alpha.$$

The inequality holds because $\varphi(\boldsymbol{y}) \leq \alpha$ and $\varphi(\boldsymbol{z}) \leq \alpha$. But these numerical considerations ensure that $\varphi(\boldsymbol{y}) = \alpha$ and $\varphi(\boldsymbol{z}) = \alpha$. As a consequence, we conclude that $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{F}$. □

The converse of Proposition 2.6.2 is false in general. In particular, a convex set can have extreme points that are not exposed. Figure 2.10 provides an example.

Remark 2.6.3 (Exposed points). For a compact convex set $\mathsf{C} \subset \mathbb{R}^d$, the set of exposed points is dense in the set of extreme points. This is a nontrivial result due to Straszewicz; see [Sch14, Thm. 1.4.7].

# Lecture 3: Extremal Representations

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*ACM 204, Fall 2018*
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*9 October 2018*

## 3.1 Agenda for Lecture 3

Extreme points are distinguished points in the boundary of a convex set. They play a fundamental role in determining the structure of the set. In this lecture, we develop two major theorems that describe how we can represent the points in a convex set in terms of the extreme points.

1. Review of extreme points, faces and exposed faces
2. Minkowski's Theorem
3. Dubins's Theorem

## 3.2 Review of extreme points, faces and exposed faces

We begin with a review of established results on the facial structure of a convex set.

### 3.2.1 Extreme points

Let $\mathsf{C} \subset \mathbb{R}^d$ be a closed, convex set. A point $\boldsymbol{x} \in \mathsf{C}$ is called an *extreme point* if $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C}$ and $\frac{1}{2}(\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x}$ together imply that $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{x}$.

In other words, an extreme point of $\mathsf{C}$ cannot be represented as the average of other points in $\mathsf{C}$. If $\boldsymbol{x} \in \mathsf{C}$ is an extreme point, then $\boldsymbol{x} \in \operatorname{relbd} \mathsf{C}$. Otherwise, $\boldsymbol{x}$ would have an open neighborhood (relative to the affine hull of $\mathsf{C}$) that is contained in $\mathsf{C}$, and we could find two points in the neighborhood that average to $\boldsymbol{x}$.

We write $\operatorname{ext} \mathsf{C}$ for the set of extreme points of $\mathsf{C}$. Let us remark that the set of extreme points does not need to have any particular topological properties. Indeed, there is an a convex set in $\mathbb{R}^3$ whose extreme points do not form a closed set!

### 3.2.2 Faces

As before, let $\mathsf{C} \subset \mathbb{R}^d$ be closed and convex. A *face* $\mathsf{F}$ of $\mathsf{C}$ is a closed, convex subset of $\mathsf{C}$ with the property that $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C}$ and $\frac{1}{2}(\mathsf{y} + \boldsymbol{z}) \in \mathsf{F}$ together imply that $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{F}$. It is convenient to write $\mathsf{F} \lhd \mathsf{C}$ for the "face of" relation.

In particular, an extreme point is a 0-dimensional face. An example of a 1-dimensional face is a side of a planar polygon in $\mathbb{R}^2$.

Technically, $\emptyset$ and $\mathsf{C}$ are also considered faces, called *improper* faces. All other faces are *proper*.

Like extreme points, proper faces are subsets of $\operatorname{relbd} \mathsf{C}$. To give further intuition, we remark that a convex set remains convex if we deprive it of a face. That is, if $\mathsf{F} \lhd \mathsf{C}$, then $\mathsf{C} \setminus \mathsf{F}$ is convex.

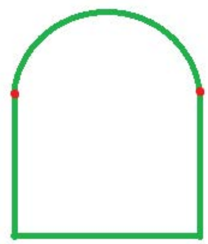

**Figure 3.1** Exposed faces shown in green, faces that are not exposed faces shown in red. Here, the non-exposed faces are two points.

### 3.2.3 Transmission of extremality

Faces have valuable transitivity properties. If $\mathsf{F} \lhd \mathsf{C}$ and $\mathsf{G} \lhd \mathsf{C}$, then $\mathsf{F} \subset \mathsf{G}$ implies that $\mathsf{F} \lhd \mathsf{G}$. This is an immediate consequence of the definition of a face.

A more subtle property is that an extreme point of a face is automatically an extreme point of the entire set.

**Proposition 3.2.1** (Transmission of Extremality)**.** *Let* $\mathsf{C} \in \mathbb{R}^d$ *be a closed, convex set, and let* $\mathsf{F} \lhd \mathsf{C}$*. Then* $\boldsymbol{x} \in \operatorname{ext} \mathsf{F}$ *implies that* $\boldsymbol{x} \in \operatorname{ext} \mathsf{C}$*.*

*Proof.* Let $\boldsymbol{x} \in \mathsf{F}$. Choose $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C}$ such that $\frac{1}{2}(\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x}$. Since $\mathsf{F}$ is a face and $\boldsymbol{x} \in \mathsf{F}$, we infer that $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{F}$. But $\boldsymbol{x} \in \operatorname{ext} \mathsf{F}$, so we must have $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{x}$. It follows that $\boldsymbol{x} \in \operatorname{ext} \mathsf{C}$. □

By the same argument, one can show that a face of a face is a face. That is, $\mathsf{F} \lhd \mathsf{G}$ and $\mathsf{G} \lhd \mathsf{C}$ ensure that $\mathsf{F} \lhd \mathsf{C}$.

### 3.2.4 Exposed Faces

Once again, let $\mathsf{C} \in \mathbb{R}^d$ be closed and convex. Let $\mathsf{H}$ be a supporting hyperplane of $\mathsf{C}$. The set $\mathsf{C} \cap \mathsf{H}$ is called an *exposed face* of $\mathsf{C}$. In particular, a 0-dimensional exposed face is called an *exposed point*.

Exposed faces are a distinct concept from faces. Nevertheless, every exposed face is a face, as proven in Proposition 6.2 of Lecture 2. The converse is not necessarily true. For a counterexample, see Figure 3.1.

By definition, each exposed point has the elegant property that it arises as the unique maximizer of a linear functional over the convex body. The following result gives a useful tool for finding extreme points.

**Proposition 3.2.2** (A Criterion for Extremity)**.** *Let* $\mathsf{C} \subset \mathbb{R}^d$ *be a closed, convex set. If a linear functional* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *achieves its maximum over* $\mathsf{C}$ *at a unique point* $\boldsymbol{x}$*, then* $\boldsymbol{x}$ *is an extreme point of* $\mathsf{C}$*.*

*Proof.* Let $\boldsymbol{x}$ be the *unique* point in $\mathsf{C}$ where $\varphi : \mathbb{R}^d \to \mathbb{R}$ achieves its maximum over $\mathsf{C}$. Consider any two points $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C}$ for which $\boldsymbol{x} = \frac{1}{2}(\boldsymbol{y} + \boldsymbol{z})$. By linearity,

$$\varphi(\boldsymbol{x}) = \frac{1}{2}\varphi(\boldsymbol{y}) + \frac{1}{2}\varphi(\boldsymbol{z}) \leq \varphi(\boldsymbol{x}).$$

Since $\varphi(\boldsymbol{y}) \leq \varphi(\boldsymbol{x})$ and $\varphi(\boldsymbol{z}) \leq \varphi(\boldsymbol{x})$, we see that $\varphi(\boldsymbol{y}) = \varphi(\boldsymbol{z}) = \varphi(\boldsymbol{x})$. But $\varphi$ achieves the value $\varphi(\boldsymbol{x})$ on $\mathsf{C}$ only at the point $\boldsymbol{x}$, so we must conclude that $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{x}$. Therefore, $\boldsymbol{x}$ is an extreme point of $\mathsf{C}$. □

Of course, the point $\boldsymbol{x} \in \mathsf{C}$ described in Proposition 3.2.2 is also an exposed point of $\mathsf{C}$. To see this directly, define $\alpha := \varphi(\boldsymbol{x})$. Introduce the hyperplane $\mathsf{H} = \{\boldsymbol{z} \in \mathbb{R}^d : \varphi(\boldsymbol{z}) = \alpha\}$. This hyperplane $\mathsf{H}$ intersects $\mathsf{C}$ at the singleton $\{\boldsymbol{x}\}$. Indeed, for all other points $\boldsymbol{y} \in \mathsf{C}$, the linear functional takes values $\varphi(\boldsymbol{y}) < \alpha$. Therefore, $\mathsf{H}$ is a supporting hyperplane of $\mathsf{C}$ containing only the point $\boldsymbol{x}$. We conclude that $\boldsymbol{x}$ is exposed.

## 3.3 Minkowski's Theorem on Extremal Representations

Minkowski's theorem on extremal representation is a major result that describes the role of extreme points in determining the structure of a convex set.

**Theorem 3.3.1** (Minkowski)**.** *Let $\mathsf{C} \subset \mathbb{R}^d$ be compact, convex, and nonempty. Then $\mathsf{C} = \operatorname{conv} \operatorname{ext} \mathsf{C}$.*

In other words, the extreme points of a compact convex set are sufficient to generate the entire set.

### 3.3.1 Consequences of Minkowski's Theorem

Before we prove Theorem 3.3.1, let us establish some corollaries.

**Corollary 3.3.2** (Existence of Extreme Points)**.** *A nonempty, compact, convex set $\mathsf{C} \subset \mathbb{R}^d$ has an extreme point.*

*Proof.* The nonempty set $\mathsf{C}$ is the convex hull of its extreme points. But the convex hull of the empty set is empty. Therefore, $\operatorname{ext} \mathsf{C}$ cannot be empty. □

The existence of extreme points is already an interesting fact. Note that compactness is necessary for Corollary 3.3.2. It is easy to construct unbounded, closed, convex sets that lack extreme points. The simplest example, perhaps, is a hyperplane.

A second consequence of Theorem 3.3.1 is that every point in a compact convex set has a parsimonious representation as a convex combination of extreme points.

**Corollary 3.3.3.** *Let $\mathsf{C} \in \mathbb{R}^d$ be compact, convex, and nonempty. If $\boldsymbol{x} \in \mathsf{C}$, then $\boldsymbol{x}$ is a convex combination of at most $d+1$ extreme points of $\mathsf{C}$.*

*Proof.* Choose $\boldsymbol{x} \in \mathsf{C}$. By Theorem 3.3.1, we have $\boldsymbol{x} \in \operatorname{conv} \operatorname{ext} \mathsf{C}$. Carathéodory's theorem (from Lecture 1) implies that $\boldsymbol{x}$ can be expressed as a convex combination of at most $d+1$ point in $\operatorname{ext} \mathsf{C}$. □

The next consequence is a key result in optimization theory.

**Corollary 3.3.4.** *Let $f : \mathsf{C} \to \mathbb{R}$ be a (finite-valued) convex function on a compact convex set $\mathsf{C} \subset \mathbb{R}^d$. Then $f$ achieves its maximum at an extreme point of $C$.*

*Proof.* Every finite-valued convex function $f$ is continuous (as we will discuss in the next lecture), so it achieves its maximum value on the compact set $\mathsf{C}$.

Let $\boldsymbol{x}_\star \in \mathsf{C}$ be a point where $f$ is maximized. By Corollary 3.3.3, we can write

$$\boldsymbol{x}_\star = \sum\nolimits_{i=1}^{d+1} \lambda_i \boldsymbol{z}_i \quad \text{where} \quad \boldsymbol{z}_i \in \operatorname{ext} \mathsf{C} \text{ and } \boldsymbol{\lambda} \in \Delta_{d+1}.$$

Jensen's inequality implies that

$$f(\boldsymbol{x}_\star) \leq \sum\nolimits_{i=1}^{d+1} \lambda_i f(\boldsymbol{z}_i) \leq \max\{f(\boldsymbol{z}_i) : i = 1, 2, 3, \dots, d+1\} =: f(\boldsymbol{z}_\star)$$

But $\boldsymbol{x}_\star$ is a maximizer of $f$ on the set $\mathsf{C}$, so the point $\boldsymbol{z}_\star \in \operatorname{ext} \mathsf{C}$ is also a maximizer. □

### 3.3.2 Proof of Theorem 3.3.1

Let us establish Minkowski's theorem. The argument applies induction on the dimension of the compact convex set.

First, let $\dim \mathsf{C} = 0$. In this case, the set is a singleton: $\mathsf{C} = \{\boldsymbol{x}\}$. The point $\boldsymbol{x}$ is obviously an extreme point of $\mathsf{C}$. It is evident that $\mathsf{C} = \operatorname{conv} \operatorname{ext} \mathsf{C}$.

Next, suppose that we have established Minkowski's theorem for compact, convex sets with dimension at most $d-1$. Let $\mathsf{C} \subset \mathbb{R}^d$ be a compact convex set, and assume that $\dim \mathsf{C} = d$. Fix a point $\boldsymbol{x} \in \mathsf{C}$. There are two possibilities to consider:

1. Suppose that $\boldsymbol{x} \in \operatorname{relbd} \mathsf{C}$. Then there exists a supporting hyperplane $\mathsf{H}$ to the set $\mathsf{C}$ at the point $\boldsymbol{x}$, by Theorem 5.2 in Lecture 2. Define the exposed face $\mathsf{F} = \mathsf{C} \cap \mathsf{H}$, which is also a face of $\mathsf{C}$. The dimension of $\mathsf{F}$ is at most $d-1$, because $\dim \mathsf{H} = d-1$. Since $\dim \mathsf{F} < d$, the inductive hypothesis implies that $\mathsf{F} = \operatorname{conv} \operatorname{ext} \mathsf{F}$. Proposition 3.2.1 states that $\operatorname{ext} \mathsf{F} \subset \operatorname{ext} \mathsf{C}$, so that $\mathsf{F} \subset \operatorname{conv} \operatorname{ext} \mathsf{C}$. Since $\boldsymbol{x} \in \mathsf{F}$, we determine that $\boldsymbol{x} \in \operatorname{conv} \operatorname{ext} \mathsf{C}$.
2. Otherwise, $\boldsymbol{x} \in \operatorname{relint} \mathsf{C}$. Since $\dim \mathsf{C} > 0$, there exists a point $\boldsymbol{y} \in \mathsf{C}$ that is different from $\boldsymbol{x}$. The two points generate a line, $\operatorname{line}(\boldsymbol{x}, \boldsymbol{y})$. This line intersects $\operatorname{relbd} \mathsf{C}$ in exactly two points because $\mathsf{C}$ is compact and convex. Denote these two points as $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \operatorname{relbd} \mathsf{C}$. By case 1, the points $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \operatorname{conv} \operatorname{ext} \mathsf{C}$. Since $\boldsymbol{x} \in [\boldsymbol{z}_1, \boldsymbol{z}_2]$, it can also be written as a convex combination of $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$. We conclude that $\boldsymbol{x} \in \operatorname{conv} \operatorname{ext} \mathsf{C}$.

The induction continues, and the proof is complete.

### 3.3.3 Extensions

Minkowski's theorem is the first result in a grand theory of extremal representations. On your homework, you will prove Klee's theorem, which gives an extremal representation of an (unbounded) closed convex set in $\mathbb{R}^d$, provided the set does not contain any lines.

In fact, a version of Minkowski's theorem holds in outrageous generality.

**Theorem 3.3.5** (Krein–Milman). *Let $\mathsf{C}$ be a nonempty compact convex subset of a locally convex topological vector space. Then $\mathsf{C} = \operatorname{cl} \operatorname{conv} \operatorname{ext} \mathsf{C}$.*

In contrast with Theorem 3.3.1, the closure is required in Theorem 3.3.5. (In infinite dimensions, extreme points can behave in counterintuitive ways.)

Most of the spaces you encounter in linear analysis fall into the class of Hausdorff locally convex topological vector spaces. Compact sets in infinite-dimensional Banach spaces are a bit thin on the ground. (Indeed, an infinite-dimensional norm ball is never norm-compact!) As a consequence, it is productive to consider spaces that have weaker topologies and, therefore, admit more compact sets. Of particular importance is the dual of a Banach space, equipped with the weak-$*$ topology, because it contains some convex compact sets that are supremely natural.

**Theorem 3.3.6** (Alaoglu–Bourbaki)**.** *The closed unit ball in the dual of a Banach space is a weak-$*$ compact set.*

Theorem 3.3.6 even extends, with appropriate modifications, to a Hausdorff locally convex topological vector space.

Here is a specific example of critical importance. Let $\mathbb{X}$ be a compact Hausdorff space, and let $C(\mathbb{X})$ be the set of (real-valued) continuous functions on $\mathbb{X}$, equipped with the supremum norm. The dual $C(\mathbb{X})^*$ consists of signed regular Borel measures. Its unit ball is a convex set that is weak-$*$ compact.

**Theorem 3.3.7** (Arens–Kelley)**.** *Let $\mathbb{X}$ be a compact Hausdorff space. The extreme points of the unit ball of $C(\mathbb{X})^*$ are precisely the signed point masses $\varepsilon\delta_x$. That is, $\varepsilon \in \{\pm 1\}$, and $\delta_x(f) = f(x)$ for each $x \in \mathbb{X}$ and $f \in C(\mathbb{X})$.*

This result tells us that each probability measure on $\mathbb{X}$ is the weak-$*$ limit of (finite) convex combinations of (positive) point masses.

The Krein–Milman theorem yields corollaries of the same type that we discussed in Section 3.3.1.

1. In infinite dimensions, the mere existence of extreme points can have remarkable consequences. For example, it can be used to show that a homogeneous compact group admits a Haar measure.
2. The fact that a (lower-semicontinuous) convex function achieves its maximum at an extreme point of a compact convex set is known as Bauer's maximum principle. It plays a role in the calculus of variations.
3. Some other results that can be derived from Theorem 3.3.5 include the Stone–Weierstrass theorem, the Lyapunov convexity theorem, and the Pontryagin maximum principle.

It will come as no surprise that Corollary 3.3.3 has no immediate analog in infinite dimensions. Nevertheless, the Krein–Milman theorem does have some related consequences: We can realize each point in a compact convex set $\mathsf{C}$ as a (continuous) average of extreme points. More precisely, for each $\boldsymbol{x} \in \mathsf{C}$, there is a Borel probability measure $\mu$ supported on $\operatorname{cl}\operatorname{ext}\mathsf{C}$ with the property that

$$\varphi(\boldsymbol{x}) = \int_{\operatorname{cl}\operatorname{ext}\mathsf{C}} \varphi(\boldsymbol{z})\,\mathrm{d}\mu(\boldsymbol{z}) \quad \text{for all linear functionals } \varphi.$$

This result leads to integral representation theorems of various types. Indeed, it is common that an interesting class of (normalized) functions forms a weak-$*$ compact and convex set in an appropriate topological vector space. We can then write each function as an integral over the extreme functions. Many major results in analysis follow this template.

Here is an example of great importance in approximation theory and machine learning. A function $f : \mathbb{R} \to \mathbb{R}$ is called *positive-definite* if the matrix

$$\Big[f(x_i - x_j)\Big]_{i,j=1,\dots,n} \quad \text{is psd} \qquad \text{for each finite set } \{x_1, \dots, x_n\} \subset \mathbb{R}.$$

Bôchner's theorem states that a function with $f(0) = 1$ is positive-definite if and only if it is the Fourier transform of a probability measure:

$$f(x) = \int_{\xi \in \mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi\xi x} \,\mathrm{d}\mu(\xi). \tag{3.3.1}$$

We can obtain Bôchner's theorem from Theorem 3.3.5 if we invest the (substantial) effort to verify that the complex exponentials compose the full set of extreme points of the class of (normalized) positive-definite functions. The integral representation (3.3.1) implies that *every* positive-definite function on $\mathbb{R}$ is an average of these extreme functions.

Other results that admit similar proofs include Bernstein's theorem on completely monotone functions, Loewner's theorem on operator monotone functions, and the Lévy–Khintchine theorem on infinitely divisible distributions.

To learn more about this topic, you may want to start with Barvinok's book [Bar02, Part III]. A more complete treatment appears in Simon's work [Sim11, Chaps. 8, 9]. Simon also covers Choquet theory, which refines these ideas to their essence.

Remark 3.3.8. Incidentally, Leonidas Alaoglu worked at Lockheed in Burbank for much of his career, and he continued to participate in mathematical activities at Caltech. Our mathematics department has an annual seminar, named in his honor.

## 3.4 Dubins's Theorem on Extremal Representations

The next major result is Dubins's theorem on extremal representations, which is sometimes known as the "dual Carathéodory theorem." It states that the extreme points of an affine slice of a convex set can be expressed parsimoniously as a convex combination of the extreme points of the set.

**Theorem 3.4.1** (Dubins, 1962)**.** *Let* $\mathsf{C} \subset \mathbb{R}^d$ *be compact and convex. Let* $\mathsf{L} \subset \mathbb{R}^d$ *be an affine space with codimension* $m$*. Then each extreme point of the intersection* $\mathsf{C} \cap \mathsf{L}$ *is a convex combination of at most* $m + 1$ *extreme points of* $\mathsf{C}$*.*

Affine slices of convex sets are often encountered in the field of convex optimization. Indeed, a convex program in standard form maximizes a linear function over the intersection of an affine space (determined by linear equality constraints) and a compact convex set. The objective function achieves its maximum at an extreme point, and Dubins's theorem tells us that this extreme point has a simple representation in terms of the extreme points of the compact convex set. This fact gives us a powerful tool for analyzing the solutions of a convex program.

### 3.4.1 Proof of Theorem 3.4.1

Let us establish Dubins's theorem. Define the set $\mathsf{E} := \mathsf{C} \cap \mathsf{L}$ and fix a point $\boldsymbol{x} \in \mathsf{E}$. By Minkowski's Theorem 3.3.1, we know that $\boldsymbol{x} \in \operatorname{conv} \operatorname{ext} \mathsf{C}$. Carathéodory's theorem ensures that the point $\boldsymbol{x}$ lies in the relative interior of the convex hull of an *affinely independent*

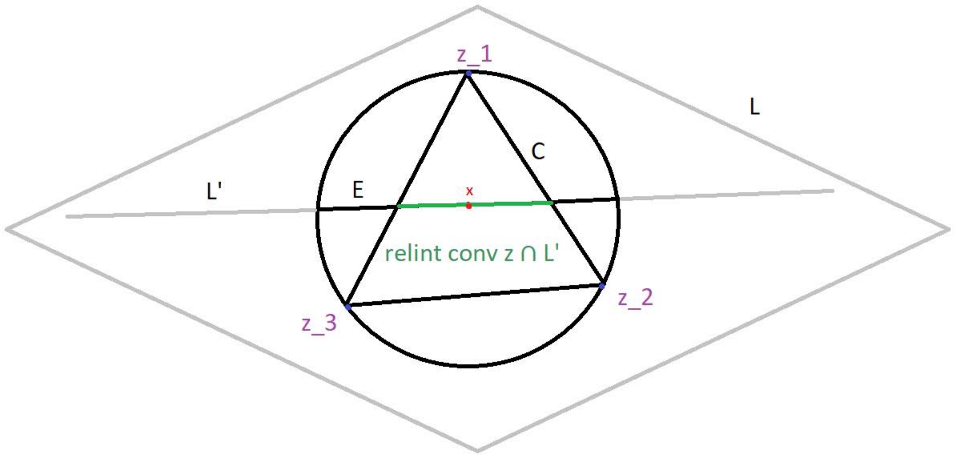


**Figure 3.2** (Proof of Dubins's theorem). The point $\boldsymbol{x}$ is captured inside an open segment inside the set $\mathsf{E}$, so it cannot be an extreme point of $\mathsf{E}$.

set $\mathsf{Z} = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_k\}$, where each $\boldsymbol{z}_i \in \operatorname{ext} \mathsf{C}$. By independence, the affine hull, $\operatorname{aff} \mathsf{Z}$, has dimension $k - 1$.

Now, assume $k - 1 > m$. It follows that $\dim \mathsf{Z} + \dim \mathsf{L} > d$. By dimension counting, the intersection of the affine spaces, $(\operatorname{aff} \mathsf{Z}) \cap \mathsf{L}$, contains a line $\mathsf{L}'$ that contains the point $\boldsymbol{x}$. (It may be easier to visualize this step if you translate the point $\boldsymbol{x}$ to the origin, in which case $\operatorname{aff} \mathsf{Z}$ and $\mathsf{L}$ are linear subspaces.) By construction,

$$\boldsymbol{x} \in (\operatorname{relint} \operatorname{conv} \mathsf{Z}) \cap \mathsf{L}' \subset \mathsf{C} \cap \mathsf{L} = \mathsf{E}.$$

We have used the facts that $\operatorname{conv} \mathsf{Z} \subset \mathsf{C}$ and that $\mathsf{L}' \subset \mathsf{L}$. This expression captures $\boldsymbol{x}$ in an open segment $(\operatorname{relint} \operatorname{conv} \mathsf{Z}) \cap \mathsf{L}'$ contained in $\mathsf{E}$. Therefore, $\boldsymbol{x}$ cannot be an extreme point of $\mathsf{E}$. See Figure 3.2 for an illustration of the geometry.

By contraposition, we conclude that $\boldsymbol{x} \in \operatorname{ext} \mathsf{E}$ implies that $k - 1 \leq m$. In other words, an extreme point of $\mathsf{E}$ can be written as a convex combination of no more than $m + 1$ extreme points of $\mathsf{C}$.

### 3.4.2 Extensions

Surprisingly, Dubins's theorem does not depend heavily on topology. We say that a set $\mathsf{C}$ in a linear space is *algebraically compact* if the intersection of set $\mathsf{C}$ with a line is always a closed bounded segment (possible empty or a point).

**Theorem 3.4.2** (Dubins, in general)**.** *Let $\mathsf{C}$ be an algebraically compact subset of a linear space, and let $\mathsf{L}$ be an affine space with codimension $m$. Each extreme point of the intersection $\mathsf{C} \cap \mathsf{L}$ can be written as a convex combination of at most $m + 1$ extreme points of $\mathsf{C}$.*

The proof of Theorem 3.4.2 is only a little more difficult than the proof of Theorem 3.4.1. The key is to establish the special case where $\mathsf{L}$ has codimension one, which follows from a direct geometric argument. We can iterate this basic result instead of invoking Minkowski's theorem, as we did in the proof of Theorem 3.4.1. See Barvinok's book [Bar02, Sec. III.9] for details.

Dubins's Theorem 3.4.2 has striking applications to probability. Indeed, consider the set $\Delta(\mathbb{X})$ of probability measures on a compact Hausdorff space $\mathsf{X}$. The set $\Delta(\mathbb{X})$ is algebraically compact, and its extreme points are the point masses $\delta_x$ for $x \in \mathbb{X}$, by the Arens–Kelley Theorem 3.3.7. Slices of the set $\Delta(\mathbb{X})$ arise when we constrain moments of a probability measure.

For concreteness, consider the case where $\mathbb{X} = [0,1] \subset \mathbb{R}$. The moments of a measure $\mu$ on $[0,1]$ take the form

$$\int_0^1 f(x)\,\mathrm{d}\mu(x) \quad \text{where} \quad f : [0,1] \to \mathbb{R} \text{ is a continuous function.}$$

Specific examples include the first and second polynomial moments:

$$\mu_1 = \int_0^1 x\,\mathrm{d}\mu(x) \quad \text{and} \quad \mu_2 = \int_0^1 x^2\,\mathrm{d}\mu(x)$$

Measures with fixed moments form an affine space. For instance, the measures with mean $\alpha$ form an affine space with codimension one:

$$\mathsf{L} = \left\{ \mu : \int_0^1 x\,\mathrm{d}\mu(x) = \alpha \right\} \quad \text{for } \alpha \in \mathbb{R}.$$

Dubins's Theorem 3.4.2 states that the extreme points of the intersection $\Delta([0,1]) \cap \mathsf{L}$ can be written as a convex combination of at most two extreme points of $\Delta([0,1])$, viz., a convex combination of at most two point masses.

Suppose that we wish to solve a linear maximization problem like

$$\underset{\mu \in \Delta([0,1])}{\text{maximize}} \quad \int_0^1 f(x)\,\mathrm{d}\mu(x) \quad \text{subject to} \quad \int_0^1 x\,\mathrm{d}\mu(x) = \alpha.$$

Then we can restrict our attention to those measures of the form $\bar{\tau}\delta_{x_1} + \tau\delta_{x_2}$ where $\tau \in [0,1]$. This reduction makes the challenge much less severe.

# Lecture 4: Smoothness and Convexity

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*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*11 October 2018*

## 4.1 Agenda for Lecture 4

A convex function enjoys more continuity and differentiability properties than a general function. In this lecture, we will establish the basic facts about smoothness of convex functions. We obtain a first result on the regularity of the boundary of a convex set by applying these facts to the support function. In particular, almost every supporting hyperplane to a convex body touches the body at a unique point. We will explain the implications of this result in optimization theory.

1. Review of convex functions
2. Smoothness of convex functions on $\mathbb{R}$
3. Smoothness of convex functions on $\mathbb{R}^d$
4. The boundary of a convex body

## 4.2 Review of Convex Functions

This section reviews basic definitions and terminology related to convex functions. We write $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ for the extended real numbers, equipped with the usual rules for arithmetic and order.

A convex function is defined by a geometric inequality, which states that the graph of the function lies below its secants.

**Definition 4.2.1** (Convex function)**.** A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is *convex* if

$$f(\bar{\tau}\boldsymbol{x} + \tau\boldsymbol{y}) \le \bar{\tau} f(\boldsymbol{x}) + \tau f(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d \text{ and } \tau \in [0, 1].$$

Recall that $\bar{\tau} := 1 - \tau$. A function $g : \mathbb{R}^d \to \overline{\mathbb{R}}$ is *concave* if $-g$ is convex.

Convexity is a self-improving property. Indeed, Definition 4.2.1 implies *Jensen's inequality* after a short inductive argument.

**Proposition 4.2.2** (Jensen's inequality)**.** *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be convex, and choose $k \in \mathbb{N}$. Then*

$$f\left(\sum\nolimits_{i=1}^{k} \lambda_i \boldsymbol{x}_i\right) \le \sum\nolimits_{i=1}^{k} \lambda_i f(\boldsymbol{x}_i) \quad \textit{for all } \boldsymbol{x}_i \in \mathbb{R}^d \textit{ and } \boldsymbol{\lambda} \in \Delta_k.$$

A function $f$ is called *proper* if it takes at least one finite value. In particular, a proper convex function cannot take the value $-\infty$. (This is a consequence of the convention that $(-\infty) + (+\infty) = \mathrm{NaN}$, which is treated as an incomparable quantity.) A proper convex function may take the value $+\infty$, provided that it is not identically equal to $+\infty$.

**Definition 4.2.3** (Domain)**.** The *domain* of a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is the set of points where it takes finite values:

$$\operatorname{dom} f = \{\boldsymbol{x} \in \mathbb{R}^d : f(\boldsymbol{x}) \ne \{\pm\infty\}\}.$$

Observe that the domain of a convex (or concave) function must be a convex set.

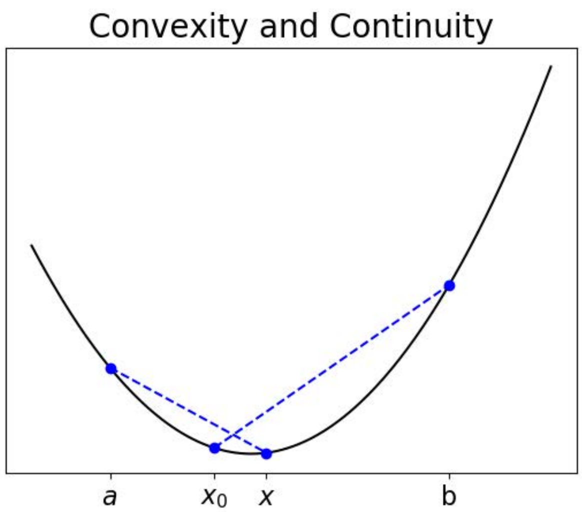


**Figure 4.1** (Convexity implies continuity). The proof of Theorem 4.3.1 compares the slopes of the two secants shown in the diagram.

## 4.3 Smoothness of Univariate Convex Functions

Let us begin with the smoothness properties of a convex function of a single real variable. We demonstrate that the function is continuous. Then we show that it has one-sided derivatives, and we prove that the function is differentiable almost everywhere.

### 4.3.1 Continuity

Our first theorem states that convexity implies continuity.

**Theorem 4.3.1** (Univariate convex functions are convex). *If $f : I \to \mathbb{R}$ is convex on an interval $I \subset \mathbb{R}$, then $f$ is continuous on* $\operatorname{int} I$.

*Proof.* Fix a point $x_0 \in \operatorname{int} I$. By definition of $\operatorname{int} I$, we can assume $x_0$ belongs to an open interval $x_0 \in (a, b) \subset \operatorname{int} I$.

For any $x \in (x_0, b)$, we can write $x = (1-\tau)x_0 + \tau b$ for some $\tau \in (0, 1)$. As $x \downarrow x_0$, the corresponding $\tau \downarrow 0$. By convexity,

$$\begin{aligned} f(x) &\le (1-\tau)f(x_0) + \tau f(b) \\ &= f(x_0) + \tau(f(b) - f(x_0)) \to f(x_0) \quad \text{as } x \downarrow x_0. \end{aligned}$$

That is, $\lim_{x\downarrow x_0} f(x) \le f(x_0)$. See Figure 4.1 for an illustration.

Along the same lines, we can write $x_0 = (1-\tau)x + \tau a$ for some $\tau \in (0, 1)$. Note that, as $x \downarrow x_0$, the corresponding $\tau \downarrow 0$. By convexity,

$$\frac{f(x_0)}{1-\tau} \le f(x) + \frac{\tau}{1-\tau} f(a).$$

As a consequence, $\lim_{x\downarrow x_0} f(x) \ge f(x_0)$. Figure 4.1 contains an illustration.

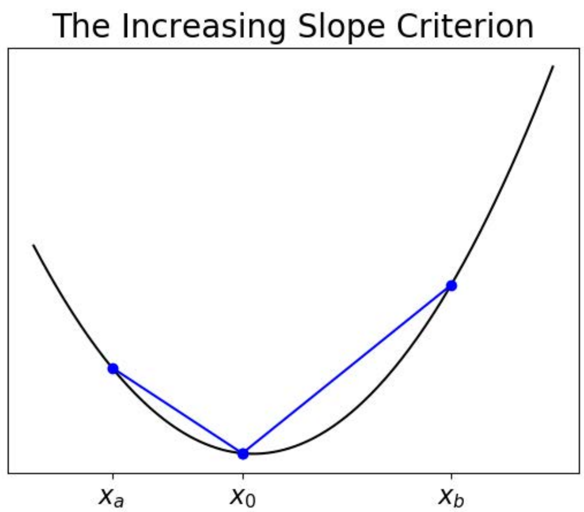


**Figure 4.2** (Increasing slope criterion). Proposition 4.3.2 states that convex functions have secants of increasing slope. In the figure above, the slope of the secant between $f(x_0)$ and $f(x)$ increases in $x$, starting out negative (for $x < x_0$, such as $x = x_a$) and becoming positive (for $x > x_0$, such as $x = x_b$).

From the last two paragraphs, it follows that $\lim_{x \downarrow x_0} f(x) = f(x_0)$. A similar argument yields $\lim_{x \uparrow x_0} f(x) = f(x_0)$. We conclude that $f$ is continuous at $x_0$. Therefore, $f$ is continuous on $\operatorname{int} I$. □

### 4.3.2 The Increasing Slope Criterion

The following proposition characterizes convex functions as those functions whose secants have increasing slope. Refer to Figure 4.2 for an illustration of the increasing slope criterion. The proof is a short algebraic exercise.

**Proposition 4.3.2** (Increasing slope criterion)**.** *The following statements are equivalent.*

1. *The function $f : I \to \mathbb{R}$ is convex on an interval $I \subset \mathbb{R}$.*
2. *For each $x_0 \in I$, the secants through $x_0$ have weakly increasing slopes. That is,*

$$x \mapsto \frac{f(x) - f(x_0)}{x - x_0} \quad \textit{is weakly increasing on } I \setminus \{x_0\}$$

### 4.3.3 Directional Derivatives

Next, we demonstrate that a convex function of a real variable has one-sided derivatives.

**Theorem 4.3.3** (Convex functions have one-sided derivatives)**.** *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$. For each $x_0 \in \operatorname{int} I$, the left derivative $\mathrm{D}^- f(x_0)$ and the right derivative*

$\mathrm{D}^+ f(x_0)$ *both exist:*

$$\mathrm{D}^- f(x_0) := \lim_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \sup_{x < x_0} \frac{f(x) - f(x_0)}{x - x_0};$$
$$\mathrm{D}^+ f(x_0) := \lim_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \inf_{x > x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

*Furthermore,* $\mathrm{D}^- f(x_0) \leq \mathrm{D}^+ f(x_0)$.

*Proof.* Select points $x < x_0 < y$. Proposition 4.3.2 provides that

$$\frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(y) - f(x_0)}{y - x_0}.$$

For fixed $y$, we obtain an upper bound on the left-hand side. Proposition 4.3.2 shows that the left-hand side is weakly increasing with $x$. Therefore, as $x \uparrow x_0$, the left-hand side converges to a limit, which allows us to define $\mathrm{D}^- f(x_0)$. Proposition 4.3.2 again ensures that this limit coincides with the supremum over $x < x_0$.

A similar argument, interchanging the role of $x$ and $y$, guarantees that $\mathrm{D}^+ f(x_0)$ exists and justifies the representation as an infimum.

Proposition 4.3.2 and the variational formulas for the left and right derivatives together ensure that $\mathrm{D}^- f(x_0) \leq \mathrm{D}^+ f(x_0)$. □

Although convexity ensures that the left and right derivatives exist, they need not coincide. For instance, the left and right derivatives of the convex function $f(x) = |x|$ satisfy $\mathrm{D}^- f(0) = -1$ and $\mathrm{D}^+ f(0) = +1$.

### 4.3.4 Differentiability

For a convex function of a real variable, the left and right derivatives are indeed equal at most points.

**Corollary 4.3.4** (Differentiability, almost everywhere). *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$. Then $f$ is differentiable almost everywhere on* $\operatorname{int} I$.

*Proof.* Repeated application of Theorem 4.3.3 yields

$$\mathrm{D}^- f(x) \leq \mathrm{D}^+ f(x) \leq \frac{f(y) - f(x)}{y - x} \leq \mathrm{D}^- f(y) \leq \mathrm{D}^+ f(y) \quad \text{when } x < y. \tag{4.3.1}$$

A first consequence of the string (4.3.1) of inequalities is that $\mathrm{D}^- f$ and $\mathrm{D}^+ f$ are both weakly increasing functions on $\operatorname{int} I$.

Here is a second consequence. Suppose that $\mathrm{D}^- f$ is continuous at a point $x_0 \in \operatorname{int} I$. Then $f$ is differentiable at $x_0$. Indeed, we can fix $y = x_0$ and take limits in (4.3.1) as $x \uparrow x_0$. Afterward, fix $x = x_0$ and take limits in (4.3.1) as $y \downarrow x_0$. Altogether, we obtain

$$\mathrm{D}^- f(x_0) = \lim_{x \uparrow x_0} \mathrm{D}^- f(x) \leq \lim_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq \mathrm{D}^- f(x_0);$$
$$\mathrm{D}^- f(x_0) = \lim_{y \downarrow x_0} \mathrm{D}^- f(y) \geq \lim_{y \downarrow x_0} \frac{f(y) - f(x_0)}{y - x_0} \geq \mathrm{D}^- f(x_0).$$

It follows that $f$ is differentiable at $x_0$ and

$$\mathrm{D}f(x_0) = \lim_{z \to x_0} \frac{f(z) - f(x_0)}{z - x_0} = \mathrm{D}^- f(x_0).$$

Likewise, continuity of $\mathrm{D}^+ f$ at $x_0$ implies differentiability of $f$ at $x_0$.

We have seen that $\mathrm{D}^- f$ is a weakly increasing function on $\operatorname{int} I$, so it can have no more than a countable number of discontinuities. Therefore, the derivative of $f$ fails to exist at no more than a countable number of points in $\operatorname{int} I$. In particular, $f$ is differentiable almost everywhere in $\operatorname{int} I$. □

## 4.4 Smoothness of Multivariate Convex Functions

We now build on results from the previous section to develop smoothness results for convex functions in $d$ dimensions.

### 4.4.1 Continuity

The first result states that, as in one dimension, a convex function on $\mathbb{R}^d$ is continuous.

**Theorem 4.4.1** (Local Lipschitz property)**.** *A convex function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is locally Lipschitz on $\operatorname{int}\operatorname{dom} f$. That is, for each $\boldsymbol{x}_0 \in \operatorname{int}\operatorname{dom} f$ and sufficiently small $\varepsilon > 0$, there exists a constant $L \in \mathbb{R}_+$ such that*

$$|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)| \leq L \cdot \|\boldsymbol{x} - \boldsymbol{x}_0\| \quad \text{for all } \boldsymbol{x} \in \mathsf{N}(\boldsymbol{x}_0; \varepsilon).$$

*In particular, $f$ is continuous on $\operatorname{int}\operatorname{dom} f$. Recall that $\mathsf{N}(\boldsymbol{x}_0; \varepsilon)$ is the open Euclidean ball of radius $\varepsilon$ centered at $\boldsymbol{x}_0$.*

*Proof.* See [Sch14, Thm. 1.5.3] or [Gru07, Thm. 2.2] for the argument. It is similar in spirit to the proof of Theorem 4.3.1, but it requires some extra technical arguments. □

### 4.4.2 Directional Derivatives

Next, we introduce the concept of a directional derivative of a multivariate function.

**Definition 4.4.2** (One-sided directional derivative)**.** Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$. For $\boldsymbol{x} \in \operatorname{int}\operatorname{dom} f$, the derivative of $f$ in the direction $\boldsymbol{u} \in \mathbb{R}^d$ is defined by the limit

$$f'(\boldsymbol{x}; \boldsymbol{u}) := \lim_{\lambda \downarrow 0} \frac{f(\boldsymbol{x} + \lambda \boldsymbol{u}) - f(\boldsymbol{x})}{\lambda}.$$

Note that $\boldsymbol{u}$ need not be a unit vector. When $\boldsymbol{u} = \mathbf{0}$, we use the convention $0/0 := 0$.

In general, a function need not have directional derivatives at any point or in any direction. The situation is more favorable for convex functions.

**Corollary 4.4.3** (Convex functions have one-sided directional derivatives)**.** *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be a convex function. Then $f'(\boldsymbol{x}; \boldsymbol{u})$ exists for all $\boldsymbol{x} \in \operatorname{int}\operatorname{dom} f$ and all $\boldsymbol{u} \in \mathbb{R}^d$.*

*Proof.* The existence of directional derivatives follows from the existence of one-sided derivatives of univariate convex functions. Indeed, the restriction of the convex function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ to the line through $\boldsymbol{x}$ and $\boldsymbol{x} + \boldsymbol{u}$ is a convex function $g : \mathbb{R} \to \overline{\mathbb{R}}$ defined by $g(t) = f(\boldsymbol{x} + t\boldsymbol{u})$. By Theorem 4.3.3, we know that $\mathrm{D}^+ g$ exists, and so $f'(\boldsymbol{x}; \boldsymbol{u})$ exists. □

As in the one-dimensional case, we can try to assemble one-sided derivatives to obtain a full derivative along a given direction.

**Definition 4.4.4** (Directional derivative)**.** We say that a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ has a *directional derivative* at a point $\boldsymbol{x} \in \operatorname{int}\operatorname{dom} f$ in the direction $\boldsymbol{u} \in \mathbb{R}^d$ if the positive and negative one-sided directional derivatives coincide:

$$f'(\boldsymbol{x}; \boldsymbol{u}) = f'(\boldsymbol{x}; -\boldsymbol{u}).$$

The situation with multivariate functions, however, is more complicated because there are many possibly directions in which we can differentiate.

### 4.4.3 Notions of Differentiability

Next, let us review several notions of multivariate differentiability. The first notion requires that a function have directional derivatives in all directions.

**Definition 4.4.5** (Gâteaux differentiability)**.** A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is *Gâteaux differentiable* at a point $\boldsymbol{x} \in \operatorname{int}\operatorname{dom} f$ if it is differentiable in every direction:

$$f'(\boldsymbol{x}; \boldsymbol{u}) = f'(\boldsymbol{x}; -\boldsymbol{u}) \quad \text{for all } \boldsymbol{u} \in \mathbb{R}^d.$$

The second notion requires that a function admits an accurate linear approximation at a point.

**Definition 4.4.6** (Fréchet differentiability)**.** A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is *Fréchet differentiable* at $\boldsymbol{x} \in \operatorname{int}\operatorname{dom} f$ if there is a linear map $\mathrm{D}f(\boldsymbol{x}) : \mathbb{R}^d \to \mathbb{R}$ such that

$$|f(\boldsymbol{x} + \boldsymbol{u}) - f(\boldsymbol{x}) - \mathrm{D}f(\boldsymbol{x})(\boldsymbol{u})| = o(\|\boldsymbol{u}\|) \quad \text{uniformly as } \boldsymbol{u} \to \boldsymbol{0}.$$

More briefly, we say that $f$ is *differentiable* at $\boldsymbol{x}$, and we refer to $\mathrm{D}f(\boldsymbol{x})$ as the *derivative*. See `https://sites.math.washington.edu/~folland/Math134/lin-approx.pdf` for the interpretation of the little-o notation in the context of a multivariate derivatives.

If a function $f$ has a derivative $\mathrm{D}f(\boldsymbol{x})$ at a point $\boldsymbol{x}$, then it has directional derivatives in all directions at $\boldsymbol{x}$. More precisely,

$$f'(x; \boldsymbol{u}) = \mathrm{D}f(\boldsymbol{x})(\boldsymbol{u}) \quad \text{for all } \boldsymbol{u} \in \mathbb{R}^d.$$

That is, Fréchet differentiability implies Gâteaux differentiability.

The converse is false in general, as you learned in multivariate calculus: There is a function on $\mathbb{R}^2$ that admits a directional derivative in each direction at the origin but does not admit a derivative at the origin. Nevertheless, for convex functions, this type of pathology is impossible.

**Theorem 4.4.7** (Gâteaux = Fréchet for convex functions)**.** *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be convex, and let $\boldsymbol{x} \in \operatorname{int}\operatorname{dom} f$. The following are equivalent:*

1. *$f$ has a directional derivative at $\boldsymbol{x}$ in each direction $\mathbf{e}_i$ for $i = 1, \dots, d$.*
2. *$f$ is Gâteaux differentiable at $\boldsymbol{x}$.*
3. *$f$ is Fréchet differentiable at $\boldsymbol{x}$.*

*Proof.* We have already discussed the facts that (3) implies (2) implies (1). It remains to check that (1) implies (3). To do so, we assemble a candidate for the derivative from the directional derivatives along the coordinates. Then we verify that this candidate satisfies the definition of the Fréchet derivative. See [Sch14, Thm. 1.5.8] for the proof. □

### 4.4.4 Differentiability of Convex Functions

We arrive now at the main event. We will demonstrate that every convex function is differentiable almost everywhere.

**Theorem 4.4.8** (Reidemeister). *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be a convex function. Then $f$ is differentiable almost everywhere in* $\operatorname{int}\operatorname{dom} f$.

*Proof.* For each $i = 1, \dots, d$, we can construct one-sided partial derivatives along the coordinate directions:

$$\mathrm{D}_i^- f(\boldsymbol{x}) := f'(\boldsymbol{x}; \mathbf{e}_i) = \lim_{\lambda \downarrow 0} \frac{f(\boldsymbol{x} + \lambda \mathbf{e}_i) - f(\boldsymbol{x})}{\lambda};$$
$$\mathrm{D}_i^+ f(\boldsymbol{x}) := f'(\boldsymbol{x}; -\mathbf{e}_i) = \lim_{\lambda \downarrow 0} \frac{f(\boldsymbol{x} - \lambda \mathbf{e}_i) - f(\boldsymbol{x})}{\lambda}.$$

Discretizing $\lambda$, we see that each one-sided partial derivative is the pointwise (in $\boldsymbol{x}$) limit of a sequence of continuous functions indexed by $\lambda$, so the one-sided partials are measurable. This is a consequence of the fact (Theorem 4.4.1) that convex functions are continuous.

For each $i = 1, \dots, d$, let us introduce the set $Y_i$ of points where the partial derivatives fail to agree:

$$Y_i := \{\boldsymbol{x} \in \operatorname{int}\operatorname{dom} f : f'(\boldsymbol{x}; \mathbf{e}_i) \neq f'(\boldsymbol{x}; -\mathbf{e}_i)\}.$$

Each set $Y_i$ is measurable because the one-sided partials are measurable. We can compute the measure of $Y_i$ by integrating its 0–1 indicator function $\mathbb{1}_{Y_i}$. By Fubini's theorem,

$$\int_{\operatorname{int}\operatorname{dom} f} \mathrm{d}\boldsymbol{x}\, \mathbb{1}_{Y_i}(\boldsymbol{x}) = \int \mathrm{d}x_d \cdots \int \mathrm{d}x_2 \underbrace{\int \mathrm{d}x_1 \mathbb{1}_{Y_i}((x_1, x_2, \dots, x_d))}_{=0} = 0.$$

Indeed, the univariate convex function $x_1 \mapsto f'((x_1, x_2, \dots, x_d); \mathbf{e}_i)$ is differentiable almost everywhere (Corollary 4.3.4). As a consequence, the restriction of $\mathbb{1}_{Y_i}$ along the first coordinate is zero almost everywhere. Thus, the inner integral vanishes.

We determine that there is zero measure on the set of "bad points" where the $i$th partial derivative fails to exist. Thus, there is zero measure on the union of bad points where any one of the partial derivatives fails to exist. In other words, $f$ has partial derivatives along each coordinate axis almost everywhere in $\operatorname{int}\operatorname{dom} f$. By Theorem 4.4.7, we conclude that $f$ has a derivative almost everywhere in $\operatorname{int}\operatorname{dom} f$. □

Remark 4.4.9 (Alexandrov's theorem). In fact, a convex function is twice differentiable almost everywhere on the interior of its domain. This theorem, due to Alexandrov, is significantly harder to prove. See [Gru07, Thm. 2.9].

## 4.5 The Boundary of a Convex Body

We have started discussing the facial structure of a convex set, and we have seen that the extreme points are sufficient to generate all of the points in the convex set. It is natural to ask how much of the boundary can be "flat" and how much is "pointy." These questions provide a motivation for studying the regularity of the boundary a convex set.

In this section, we derive one of the simplest regularity results. We will show that, in most directions, a compact convex convex set has a supporting hyperplane that touches the set at a unique point. Afterward, we develop an application of this result in optimization.

### 4.5.1 The Support Function

Let $\mathsf{C} \subset \mathbb{R}^d$ be a compact convex set. Recall that the *support function* is defined as

$$h_{\mathsf{C}}(\boldsymbol{s}) := \max\{\langle \boldsymbol{s}, \boldsymbol{x}\rangle : \boldsymbol{x} \in \mathsf{C}\} \quad \text{for } \boldsymbol{s} \in \mathbb{R}^d.$$

Intuitively, the support function takes a direction $\boldsymbol{s}$, and reports the level $\alpha \in \mathbb{R}$ of the hyperplane $\mathsf{H}_{\boldsymbol{s},\alpha}$ with (outer) normal $\boldsymbol{s}$ that supports $\mathsf{C}$.

Some of the basic properties of the support function follow instantly from the definition. First, the support function is *convex* because it is the pointwise maximum of linear functions. For the same reason, the sublevel sets of the support function are *closed*. The support function is also *positive homogeneous*:

$$h_{\mathsf{C}}(\lambda \boldsymbol{s}) = \lambda h_{\mathsf{C}}(\boldsymbol{s}) \quad \text{for } \lambda \geq 0 \text{ and } \boldsymbol{s} \in \mathbb{R}^d.$$

A positive homogeneous, convex function is also called *sublinear*.

### 4.5.2 The Subdifferential

Recall that the *subdifferential* of a convex function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ at a point $\boldsymbol{y} \in \operatorname{relint} \operatorname{dom} f$ is defined as

$$\partial f(\boldsymbol{y}) := \{\boldsymbol{u} \in \mathbb{R}^d : f(\boldsymbol{z}) - f(\boldsymbol{y}) \geq \langle \boldsymbol{u}, \boldsymbol{z} - \boldsymbol{y}\rangle \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^d\}.$$

In other words, the subdifferential contains the slopes of affine lower bounds for the function at the point $\boldsymbol{y}$. This definition generalizes the gradient inequality that holds when the convex function $f$ is differentiable at $\boldsymbol{y}$:

$$f(\boldsymbol{z}) - f(\boldsymbol{y}) \geq \mathrm{D}f(\boldsymbol{y})(\boldsymbol{z} - \boldsymbol{y}) \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^d.$$

In particular, if $f$ is differentiable at $\boldsymbol{y}$, then $\mathrm{D}f(\boldsymbol{y})$ belongs to the subdifferential $\partial f(\boldsymbol{y})$. In fact, $f$ is differentiable at $\boldsymbol{y}$ if and only if the derivative $\mathrm{D}f(\boldsymbol{y})$ is the unique element of the subdifferential.

### 4.5.3 Exposed Faces and the Support Function

We may now connect the smoothness properties of the support function with the boundary structure of a convex set. The following result follows from the definition of the support function, the definition of the subdifferential, and a short computation.

**Proposition 4.5.1** (Subdifferential of support function)**.** *The subdifferential of the support function $h_{\mathsf{C}}$ of a closed convex set $\mathsf{C} \subset \mathbb{R}^d$ takes the form*

$$\partial h_{\mathsf{C}}(\boldsymbol{s}) = \{\boldsymbol{x} \in \mathsf{C} : \langle \boldsymbol{s}, \boldsymbol{x}\rangle = h_{\mathsf{C}}(\boldsymbol{s})\} =: \mathsf{F}_{\mathsf{C}}(\boldsymbol{s}) \quad \textit{for } \boldsymbol{s} \in \mathbb{R}^d.$$

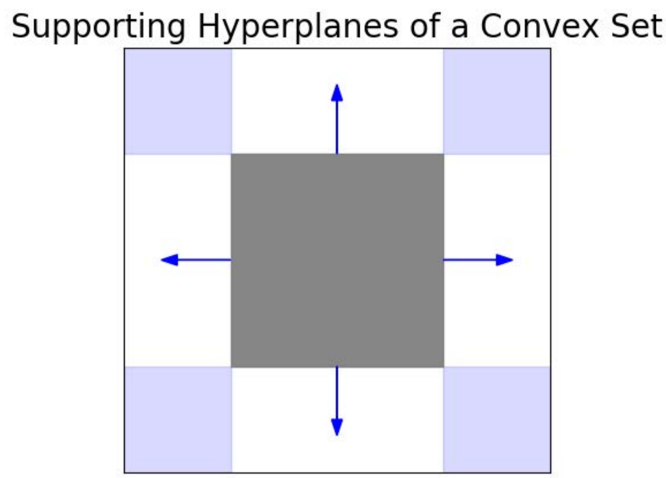


**Figure 4.3** (Exposed points in almost every direction). For almost every direction $\boldsymbol{u} \in \mathbb{S}^{d-1}$, the face of a convex set $\mathsf{C}$ (grey) exposed in the direction $\boldsymbol{u}$ consists of a single point. The dark blue arrows correspond to the four directions where the exposed face of $\mathsf{C}$ is not a point. Each vector pointing from a vertex of the square into the adjacent light blue region is a direction where the exposed face is a point (namely, the vertex).

*In other words, the subdifferential of the support function in a direction $\boldsymbol{s}$ is the face $\mathsf{F}_{\mathsf{C}}(\boldsymbol{s})$ of the set $\mathsf{C}$ that is exposed in the direction $\boldsymbol{s}$.*

As an immediate corollary, we see that the support function of $\mathsf{C}$ is differentiable at $\boldsymbol{s}$ when the face exposed in the direction $\boldsymbol{s}$ consists of a single point.

**Corollary 4.5.2** (Derivative of support function)**.** *The support function $h_{\mathsf{C}}$ of a closed convex set $\mathsf{C} \subset \mathbb{R}^d$ is differentiable at $\boldsymbol{s} \in \mathbb{R}^d$ if and only if $\mathsf{F}_{\mathsf{C}}(\boldsymbol{s})$ is an exposed* point *of $\mathsf{C}$.*

We may now invoke Reidemeister's theorem to obtain our main result. See Figure 4.3 for an illustration.

**Corollary 4.5.3** (Almost every direction exposes a point)**.** *Let $\mathsf{C} \subset \mathbb{R}^d$ be a compact convex set. For almost every $\boldsymbol{s} \in \mathbb{R}^d$, the face $\mathsf{F}_{\mathsf{C}}(\boldsymbol{s})$ exposed in the direction $\boldsymbol{s}$ is an exposed point.* A fortiori*, for almost every $\boldsymbol{u} \in \mathbb{S}^{d-1}$, the face $\mathsf{F}_{\mathsf{C}}(\boldsymbol{u})$ exposed in the direction $\boldsymbol{u}$ is an exposed point.*

*Proof.* By Theorem 4.4.8, the support function $h_{\mathsf{C}}$ is differentiable almost everywhere. By Corollary 4.5.2, the support function is differentiable at $\boldsymbol{s}$ if and only if $\mathsf{F}_{\mathsf{C}}(\boldsymbol{s})$ is an exposed point. The first conclusion follows.

We sketch a proof of the second claim. The support function is positively homogeneous, so it is differentiable at a point $\boldsymbol{s}$ if and only if it is differentiable on the entire ray $\{\lambda \boldsymbol{s} : \lambda > 0\}$. If the support function failed to be differentiable on a subset of the sphere with positive measure, the rays passing through this subset would compose a set of positive measure in $\mathbb{R}^d$. □

Remark 4.5.4 (Smoothness). If we apply Alexandrov's theorem (Remark 4.4.9) to the support function, we see that the support function is *twice* differentiable almost everywhere. This means that the level of the supporting hyperplane varies smoothly with the direction, most of the time. For further results on the smoothness of the boundary, see [Sch14, Chap. 2] or [Gru07, Chap. 5].

### 4.5.4 Application to Random Optimization

We conclude the lecture with a striking application of Corollary 4.5.3 in optimization. Suppose we want to find an extreme point of a compact convex set $\mathsf{C} \subset \mathbb{R}^d$. This is possible so long as we can optimize an arbitrary linear function over the set $\mathsf{C}$.

Indeed, let us draw a standard normal vector $\boldsymbol{g} \in \mathbb{R}^d$. The Lebesgue measure and the standard normal measure have the same null sets (i.e., sets with zero measure). Therefore, the conclusion of Reidemeister's theorem holds almost everywhere with respect to the standard normal measure. In other words, with probability one, a standard normal vector $\boldsymbol{g}$, exposes a face of $\mathsf{C}$ that is an exposed point.

Now, suppose that we solve the (random) optimization problem

$$\text{maximize} \quad \langle \boldsymbol{g}, \boldsymbol{x} \rangle \quad \text{subject to} \quad \boldsymbol{x} \in \mathsf{C}.$$

With probability one, the optimization problem has a unique solution that is an exposed point, by definition of an exposed point. Exposed points are extreme, so we conclude that the random optimization also locates an extreme point with probability one.

Suppose instead that we draw a vector $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ uniformly at random from the unit sphere and solve the optimization problem

$$\text{maximize} \quad \langle \boldsymbol{\theta}, \boldsymbol{x} \rangle \quad \text{subject to} \quad \boldsymbol{x} \in \mathsf{C}.$$

Then the unique solution is also an extreme point of $\mathsf{C}$.

Warning 4.5.5 (Collecting coupons). This method is not an effective way to find all the exposed points of a convex set. Even if we find each of $n$ exposed points with equal probability, it will require about $n \log n$ repetitions to find all $n$ because of the coupon collector problem. Moreover, certain exposed points may appear with very low probability, and it is very hard to find them by random sampling. See [GLS88, Sec. 6.5] for a discussion about methods for finding vertices of polyhedra.

# Lecture 5: Polarity and the Weyl–Minkowski Theorem

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*ACM 204, Fall 2018*
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*16 October 2018*

## 5.1 Agenda for Lecture 5

Polarity is an essential tool for understanding the structure of convex sets. In this lecture, we present the Bipolar Theorem. Then we introduce polytopes and polyhedra, which are two basic classes of convex sets. Last, we prove the Weyl–Minkowski theorem, which gives an equivalence between polytopes and bounded polyhedra.

1. Polarity
2. Polytopes and polyhedra
3. The Weyl–Minkowski Theorem

## 5.2 Polarity

In convex geometry, duality exchanges points with hyperplanes. It provides a mechanism for translating facts about points to new facts about hyperplanes (and conversely). Polarity is a duality operation for convex sets that exchanges vertices with facets, and it provides a valuable tool for understanding the structure of a convex set.

### 5.2.1 Polar Sets

To begin, we introduce the notion of the polar of a set. Then we relate the polar to the original set. In particular, we show that polarity is an involution on certain classes of convex sets. That is, applying polarity twice sometimes returns the original set.

**Definition 5.2.1** (Polar Set)**.** Let $\mathsf{A} \subset \mathbb{R}^d$ be a nonempty set. The polar set of $\mathsf{A}$ is

$$\mathsf{A}^\circ := \{\boldsymbol{s} \in \mathbb{R}^d : \langle \boldsymbol{s}, \boldsymbol{x} \rangle \leq 1 \text{ for all } \boldsymbol{x} \in \mathsf{A}\}.$$

The definition immediately yields some basic facts about the polar operation.

**Proposition 5.2.2** (Basic properties of polarity)**.** *Let $\mathsf{A} \subset \mathbb{R}^d$ be nonempty set. Then the polar $\mathsf{A}^\circ$ is a closed convex set that contains the origin $\mathbf{0}$.*

*Proof.* The appearance of the origin is evident. The polar is an intersection of closed halfspaces, so it must be a closed convex set. □

Polarity is probably familiar to you already because it is the precise relationship between the unit ball of a norm and the unit ball of the dual norm. Figure 5.1 illustrates the polarity relation for a few specific examples. Here is a catalog:

- If $\mathsf{C} = \varepsilon \cdot \mathsf{B}_d$ for $\varepsilon > 0$, then $\mathsf{C}^\circ = \varepsilon^{-1} \cdot \mathsf{B}_d$.
- If $\mathsf{C}$ is the $\ell_1^d$ unit ball, then $\mathsf{C}^\circ$ is the $\ell_\infty^d$ unit ball.
- If $\mathsf{C}$ is the $\ell_\infty^d$ unit ball, then $\mathsf{C}^\circ$ is the $\ell_1^d$ unit ball.

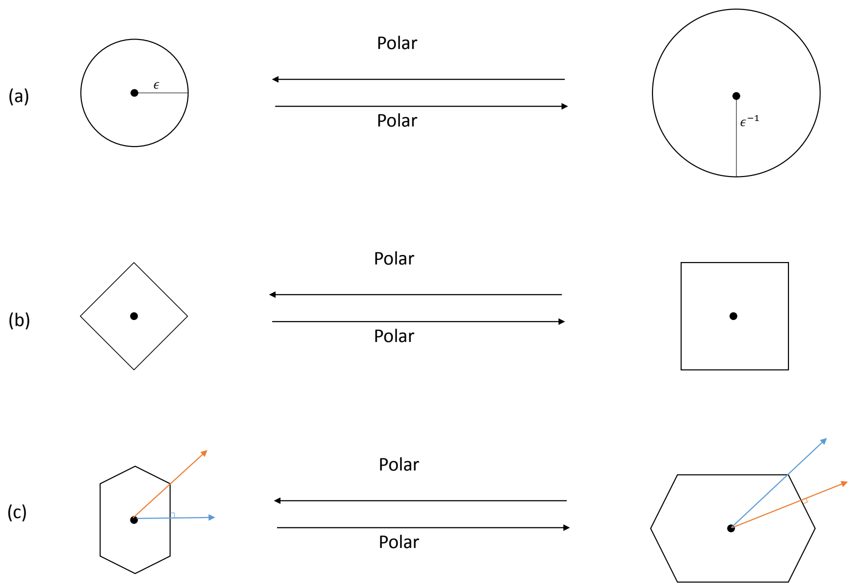



**Figure 5.1** (Dual of Norm Balls). From top to bottom: (a) The polar of the Euclidean ball with radius $\varepsilon$ is the Euclidean ball with radius $\varepsilon^{-1}$. (b) The polar of the $\ell_\infty$ ball is the $\ell_1$ ball and vice versa. (c) The polar of a hexagonal unit ball in $\mathbb{R}^2$ switches vectors that intersect vertices and vectors orthogonal to facets.

- A hexagonal unit ball in $\mathbb{R}^2$ and its polar exchange vectors that intersect vertices with vectors that are orthogonal to facets.
- In general, if $\mathsf{C}$ is a norm ball, then $\mathsf{C}^\circ$ is the unit ball of the dual norm.

On the basis of these simple examples, we can make the following geometric observations about the action of polarity:

1. Vertices are exchanged with facets.
2. Directions of elongation are exchanged with directions of compression.
3. In a finite-dimensional space, the dual of the dual of a norm ball is the norm ball itself.
4. More generally, applying polarity twice appears to undo its effect.

Let us develop a more detailed understanding of how polarity behaves. This analysis will justify some of the foregoing claims more completely. First, we note that polarity reverses the order on sets.

**Proposition 5.2.3** (Polarity reverses inclusion)**.** *If* $\mathsf{A} \subset \mathsf{B}$*, then* $\mathsf{B}^\circ \subset \mathsf{A}^\circ$*.*

*Proof.* This result is an immediate consequence of Definition 5.2.1. □

Next, we show that applying polarity twice leads to a set at least as large as the one we started with.

**Proposition 5.2.4** (The double polar is nondecreasing)**.** *For* $\mathsf{A} \subset \mathbb{R}^d$*, we have* $\mathsf{A} \subset (\mathsf{A}^\circ)^\circ := \mathsf{A}^{\circ\circ}$

*Proof.* Fix a point $\boldsymbol{x} \in \mathsf{A}$. For each $\boldsymbol{s} \in \mathsf{A}^\circ$, we have $\langle \boldsymbol{x}, \boldsymbol{s} \rangle = \langle \boldsymbol{s}, \boldsymbol{x} \rangle \leq 1$. Therefore, $\boldsymbol{x} \in \mathsf{A}^{\circ\circ}$. □

The reverse of the inclusion in Proposition 5.2.4 can fail. For example, suppose that $\mathsf{A} \subset \mathbb{R}^d$ is finite set of points. It can be shown that $\mathsf{A} \subsetneq \operatorname{conv} \mathsf{A} = \mathsf{A}^{\circ\circ}$.

### 5.2.2 The Bipolar Theorem

Although the reverse of the inclusion in Proposition 5.2.4 can fail, it can only fail for trivial reasons. If the initial set has the properties required of a polar set, as outlined in Proposition 5.2.2, then the double polar returns the original set.

**Theorem 5.2.5** (Bipolar Theorem)**.** *If* $\mathsf{C} \subset \mathbb{R}^d$ *is a closed convex set that contains the origin, then* $\mathsf{C}^{\circ\circ} = \mathsf{C}$*.*

*Proof.* By Proposition 5.2.4, it suffices to check that $\mathsf{C}^{\circ\circ} \subset \mathsf{C}$. The strategy is to use a contrapositive argument: For a point $\boldsymbol{z} \notin \mathsf{C}$, we will show that $\boldsymbol{z} \notin \mathsf{C}^{\circ\circ}$ using the separation theorem. The proof goes as follows.

Since $\mathsf{C}$ is closed and convex, we can properly separate $\boldsymbol{z}$ from $\mathsf{C}$. In other words, there exists a nonzero $\boldsymbol{s} \in \mathbb{R}^d$ such that $\langle \boldsymbol{s}, \boldsymbol{z} \rangle > \alpha$ and $\sup_{\boldsymbol{x} \in \mathsf{C}} \langle \boldsymbol{s}, \boldsymbol{x} \rangle < \alpha$. Since $\mathbf{0} \in \mathsf{C}$, we must have $\alpha > 0$. As a consequence, for each $\boldsymbol{x} \in \mathsf{C}$, it holds that $\langle \boldsymbol{s}/\alpha, \boldsymbol{x} \rangle \leq 1$. That is, $\boldsymbol{s}/\alpha \in \mathsf{C}^\circ$. However, $\langle \boldsymbol{z}, \boldsymbol{s}/\alpha \rangle = \langle \boldsymbol{s}/\alpha, \boldsymbol{z} \rangle > 1$. We conclude that $\boldsymbol{z} \notin \mathsf{C}^{\circ\circ}$. □

We can express Theorem 5.2.5 as the statement that polarity is an order-reversing involution on the class of closed convex sets that contain the origin. In fact, polarity has a similar behavior for several smaller classes of convex sets. To establish one of these results, we need another easy proposition.

**Proposition 5.2.6** (Duality of interior origin and boundedness)**.** *If* $\mathbf{0} \in \operatorname{int} \mathsf{A}$*, then* $\mathsf{A}^\circ$ *is bounded. If* $\mathsf{A}$ *is bounded, then* $\mathbf{0} \in \operatorname{int} \mathsf{A}^\circ$*.*

*Proof.* Let us establish the first statement. For some $\varepsilon > 0$, we have $\varepsilon \cdot \mathsf{B}_d \subset \mathsf{A}$. Since polarity reverses inclusion, $\mathsf{A}^\circ \subset \varepsilon^{-1} \cdot \mathsf{B}_d$.

We turn to the second statement. For some $\varrho > 0$, we have $\mathsf{A} \subset \varrho \cdot \mathsf{B}_d$. Since polarity reverses inclusion, $\varrho^{-1} \cdot \mathsf{B}_d \subset \mathsf{A}^\circ$. Therefore, $\mathbf{0} \in \operatorname{int} \mathsf{A}^\circ$. □

From Theorem 5.2.5 and Proposition 5.2.6, we conclude that polarity is an order-reversing involution on the class of compact convex sets that contain the origin in their interior.

Remark 5.2.7 (Convex cones). Polarity is also an order-reversing involution on the class of closed convex cones. This is an exercise.

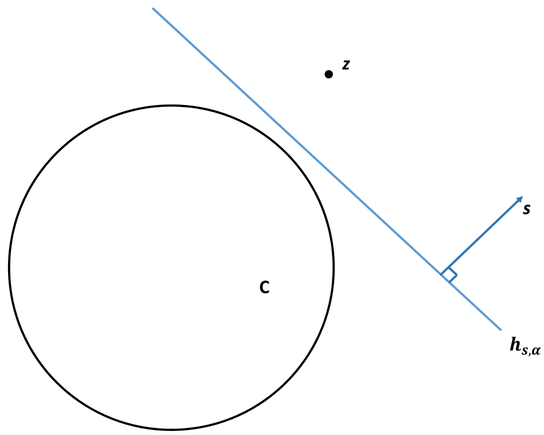



**Figure 5.2** (Proof of the bipolar theorem). For a point $z \notin \mathsf{C}$, we can properly separate the point $z$ from the set $\mathsf{C}$. The separating hyperplane yields a vector $s/\alpha \in \mathsf{C}^\circ$. The point $z$ cannot belong to the double polar $\mathsf{C}^{\circ\circ}$ because of the separation.

## 5.3 Polytopes and Polyhedra

Definitions of polytopes and polyhedra are presented in this section. We highlight the key difference between the two.

**Definition 5.3.1** (Polytope)**.** A polytope is the convex hull of a finite set of points.

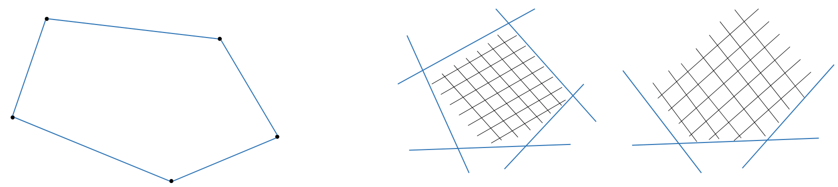
**Definition 5.3.2** (Polyhedron)**.** A polyhedron is the finite intersection of closed halfspaces.

See Figure 5.3 for an illustration of some polytopes and polyhedra. Note that a polytope is always bounded, while a polyhedron may be bounded or unbounded. Otherwise, they appear to be strikingly similar.

According to Minkowski's extremal representation theorem, a polytope has a finite number of extreme points. Moreover, every set with a finite number of extreme points is a polytope. Later, we will see that a polyhedron has a finite number of faces, and every set with a finite number of faces is a polyhedron.

## 5.4 Weyl–Minkowski Theorem

We may suspect that polytopes and polyhedra have something in common, but they are apparently not the same objects because polyhedra can be unbounded. In fact, this is the only obstacle to obtaining an equivalence. The Weyl–Minkowski theorem asserts that the class of polytopes coincides with the class of *bounded* polyhedra.



**(a)** A polytope **(b)** Two polyhedra

**Figure 5.3** (Polytopes and Polyhedrons). A polytope is the convex hull of a finite point set. A polyhedron is the intersection of a finite number of halfspaces. A polytope is always bounded, but a polyhedron can be bounded or unbounded.

**Theorem 5.4.1** (Weyl–Minkowski). *We have the following implications.*

1. *A bounded polyhedron is a polytope.*
2. *A polytope is a bounded polyhedron.*

The two parts require separate arguments. We will establish the first statement using a direct characterization of the extreme points of a polyhedron. Afterward, we will develop the second statement by invoking polarity and then applying the first statement. This is a remarkable strategy: polarity gives us a striking conclusion, almost for free.

### 5.4.1 Characterization of Vertices

The proof of the first implication in Theorem 5.4.1 is based on a characterization the extreme points of a polyhedron. In this context, an extreme point is commonly referred to as a *vertex*.

**Proposition 5.4.2** (Vertices of a polyhedron). *Consider the polyhedron*

$$\mathsf{P} := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s}_i, \boldsymbol{x} \rangle \leq \alpha_i \quad \text{for all } i = 1, 2, 3, \dots, n\}.$$

*For a point $\boldsymbol{x} \in \mathsf{P}$, introduce the set of constraints that are active at $\boldsymbol{x}$:*

$$I(\boldsymbol{x}) := \{i : \langle \boldsymbol{s}_i, \boldsymbol{x} \rangle = \alpha_i\}.$$

*If $\boldsymbol{x}$ is a vertex of $\mathsf{P}$, then $\operatorname{lin}\{\boldsymbol{s}_i : i \in I(\boldsymbol{x})\} = \mathbb{R}^d$.*

*Proof.* Let $\boldsymbol{x} \in \mathsf{P}$. Suppose that $\boldsymbol{x}$ is a vertex of $\mathsf{P}$, but $\operatorname{lin}\{\boldsymbol{s}_i : i \in I(\boldsymbol{x})\} \neq \mathbb{R}^d$. Then there exists a nonzero point $\boldsymbol{y} \in \mathbb{R}^d$ such that $\langle \boldsymbol{s}_i, \boldsymbol{y} \rangle = 0$ for all $i \in I(\boldsymbol{x})$.

For a parameter $\varepsilon > 0$, define $\boldsymbol{x}_+ := \boldsymbol{x} + \varepsilon \boldsymbol{y}$ and $\boldsymbol{x}_- := \boldsymbol{x} - \varepsilon \boldsymbol{y}$. It is clear that $\boldsymbol{x} = \frac{1}{2}(\boldsymbol{x}_+ + \boldsymbol{x}_-)$, and both points $\boldsymbol{x}_\pm$ are different from $\boldsymbol{x}$ because the vector $\boldsymbol{y}$ is nonzero.

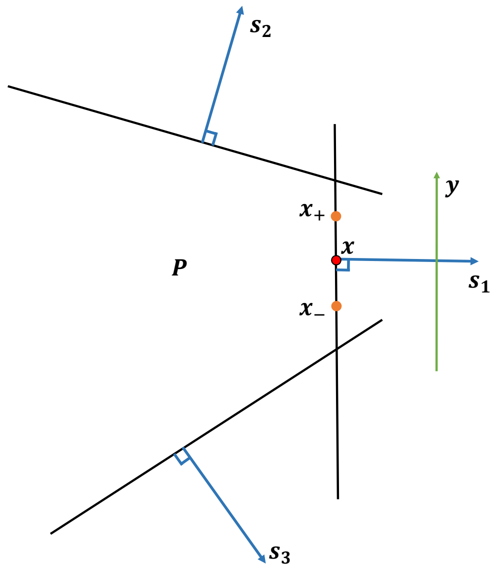



**Figure 5.4** (Proof of Proposition 5.4.2). If $\boldsymbol{x}$ is a vertex of $\mathsf{P}$, then $\mathbb{R}^d$ is spanned by the normal vectors $\boldsymbol{s}_i$ to the constraints $i \in I(\boldsymbol{x})$ that are active at $\boldsymbol{x}$.

Next, observe that $\boldsymbol{x}_\pm \in \mathsf{P}$ as soon as $\varepsilon$ is sufficiently small. Indeed, the inactive constraints $i \notin I(\boldsymbol{x})$ are finite in number, so we have a bound $\langle \boldsymbol{s}_i, \boldsymbol{x} \rangle \leq \alpha_i - \delta$ with $\delta > 0$ for all $i \notin I(\boldsymbol{x})$. As a consequence, for very small $\varepsilon$,

$$\begin{aligned} \langle \boldsymbol{s}_i, \boldsymbol{x}_\pm \rangle &= \langle \boldsymbol{s}_i, \boldsymbol{x} \rangle \pm \varepsilon \langle \boldsymbol{s}_i, \boldsymbol{y} \rangle = \alpha_i \quad \text{for all } i \in I(\boldsymbol{x}); \\ \langle \boldsymbol{s}_i, \boldsymbol{x}_\pm \rangle &= \langle \boldsymbol{s}_i, \boldsymbol{x} \rangle \pm \varepsilon \langle \boldsymbol{s}_i, \boldsymbol{y} \rangle < \alpha_i \quad \text{for all } i \notin I(\boldsymbol{x}). \end{aligned}$$

We conclude that $\boldsymbol{x}$ is not an extreme point of $\mathsf{P}$. See Figure 5.4 for an illustration of the argument. □

**Fact 5.4.3.** *The converse of Proposition 5.4.2 also holds true. That is, if $\mathbb{R}^d$ is span of the normals $\boldsymbol{s}_i$ to the active constraints $i \in I(\boldsymbol{x})$, then the point $\boldsymbol{x}$ is a vertex of $\mathsf{P}$.*

### 5.4.2 Proof of the Weyl–Minkowski Theorem

We are now prepared to prove the Weyl–Minkowski theorem.

#### A Bounded Polyhedron is a Polytope

Consider a bounded polyhedron

$$\mathsf{P} := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s}_i, \boldsymbol{x} \rangle \leq \alpha_i \quad \text{for all } i = 1, 2, 3, \dots, m\}.$$

The bounded polyhedron $\mathsf{P}$ is a compact convex set. Therefore, Minkowski's theorem tells us that $\mathsf{P}$ is the convex hull of its vertices. We must verify that the total number of vertices is finite.

Each vertex $\boldsymbol{x}$ of $\mathsf{P}$ induces a set $I(\boldsymbol{x})$ of active constraints, which compose a linear system:

$$\langle \boldsymbol{s}_i, \boldsymbol{x} \rangle = \alpha_i \quad \text{for each } i \in I(\boldsymbol{x}).$$

By Proposition 5.4.2, the family $\{\boldsymbol{s}_i : i \in I(\boldsymbol{x})\}$ spans $\mathbb{R}^d$. So the linear system has full rank, and the distinguished vertex $\boldsymbol{x}$ is the unique solution of the linear system.

In other words, we have constructed an injection from vertices of the polyhedron into the family of subsets of constraints. Since there are at most $2^m$ subsets of constraints, the polyhedron has at most $2^m$ vertices.

Remark 5.4.4 (Vertex counting). In fact, $\mathsf{P}$ can have at most $\binom{m}{d}$ vertices. Indeed, we can identify each set $\{\boldsymbol{s}_i : i \in I\}$ of normal vectors that spans $\mathbb{R}^d$ with the subset that contains the first $d$ vectors that form a basis (using the prespecified order $\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_m$). This observation gives an injection from the set of vertices into a family of $\binom{m}{d}$ sets of constraints, or fewer.

#### A Polytope is a Bounded Polyhedron

We use polarity to translate the question about polytopes to a question about polyhedra, and then we apply the claim that we just established about polyhedra.

Consider the polytope

$$\mathsf{P} := \operatorname{conv}\{\boldsymbol{x}_1, ...\boldsymbol{x}_n\} \subset \mathbb{R}^d.$$

The polytope $\mathsf{P}$ is automatically bounded, so the challenge is to show that it admits a representation as an intersection of halfspaces. Without loss of generality, we can assume $\dim \mathsf{P} = d$ and $\mathbf{0} \in \operatorname{int} \mathsf{P}$. The polar set takes the form

$$\mathsf{P}^\circ = \{\boldsymbol{s} \in \mathbb{R}^d : \langle \boldsymbol{s}, \boldsymbol{x}_i \rangle \leq 1 \quad \text{for all } i = 1, 2, 3, \dots, n\}.$$

Since $\mathsf{P}$ contains the origin in its interior, Proposition 5.2.6 shows that the polar $\mathsf{P}^\circ$ is a bounded polyhedron. Now, the first part of the Weyl–Minkowski theorem implies that $\mathsf{P}^\circ$ is a polytope:

$$\mathsf{P}^\circ = \operatorname{conv}\{\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_m\}.$$

Polarizing again,

$$\mathsf{P}^{\circ\circ} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{s}_i \rangle \leq 1 \quad \text{for } i = 1, 2, 3, \dots, m\}.$$

But $\mathsf{P}$ is a compact convex set that contains the origin, so Theorem 5.2.5 ensures that $\mathsf{P}^{\circ\circ} = \mathsf{P}$. We have represented $\mathsf{P}$ as a finite intersection of halfspaces. In other words, $\mathsf{P}$ is a polyhedron.

# Lecture 6: Facial Decomposition

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*ACM 204, Fall 2018*
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*18 October 2018*

## 6.1 Agenda for Lecture 6

The faces of a convex set make up the boundary, and they are a basic tool for understanding the structure of the set. In this lecture, we will show that the relative interiors of the faces partition the entire set. Then we will introduce the normal cone of a face, which contains all of the vectors that are orthogonal to the vectors in a face. We can develop a fundamental decomposition of Euclidean space in terms of the relative interiors of faces and in terms of normal cones. After these generalities, we will focus our attention on the class of polytopes. We will see that all faces of a polytope are exposed, and we will show that the dimension of a face and its normal cone are complementary.

1. Facts about faces, old and new
2. Normal cones
3. The tiling induced by a convex set
4. Proper faces of a polytope are exposed
5. Structure of normal cones of polytopes

## 6.2 Facts about faces

We begin with a reminder about some of the properties of the faces of a convex set, and then we continue with some new developments.

**Fact 6.2.1** (What we already know about faces)**.** *Suppose* $\mathsf{C} \subset \mathbb{R}^d$ *is a closed convex set, and let* $\mathsf{F} \subset \mathsf{C}$ *be a face of* $\mathsf{C}$*. The face has the following properties.*

1. *If* $\mathsf{F}$ *is a proper face of* $\mathsf{C}$*, then* $\mathsf{F}$ *is contained in* $\operatorname{relbd} \mathsf{C}$*.*
2. *If* $\mathsf{F}'$ *is a face of* $\mathsf{C}$*, then* $\mathsf{F} \cap \mathsf{F}'$ *is also a face of* $\mathsf{C}$*.*
3. *If* $\mathsf{G}$ *is a face of* $\mathsf{F}$*, then* $\mathsf{G}$ *is a face of* $\mathsf{C}$*.*
4. *If* $\mathsf{F}'$ *is an exposed face of* $\mathsf{C}$*, then* $\mathsf{F}'$ *is a face of* $\mathsf{C}$*.*

Next, we show that distinct faces have disjoint relative interiors. Furthermore, the relative interiors of the faces partition the entire set.

**Proposition 6.2.2** (Faces partition a convex set)**.** *Suppose* $\mathsf{C} \subset \mathbb{R}^d$ *is a closed convex set. Then*

1. *If* $\mathsf{F}_1, \mathsf{F}_2$ *are distinct faces of* $\mathsf{C}$*, then* $\operatorname{relint}(\mathsf{F}_1) \cap \operatorname{relint} \mathsf{F}_2 = \emptyset$*.*
2. *Each point* $\boldsymbol{x} \in \mathsf{C}$ *belongs to the relative interior of a unique face of* $\mathsf{C}$*.*

*Proof.* We prove the first statement by contradiction. Suppose that $\boldsymbol{x} \in \operatorname{relint} \mathsf{F}_1 \cap \operatorname{relint} \mathsf{F}_2$. Since $\mathsf{F}_1$ and $\mathsf{F}_2$ are distinct, we can assume that there is a point $\boldsymbol{y} \in \mathsf{F}_1 \backslash \mathsf{F}_2$. (Otherwise, interchange the two sets and proceed.)

Because $\boldsymbol{x} \in \operatorname{relint} \mathsf{F}_1$ and $\boldsymbol{y} \in \mathsf{F}_1$, we can find a third point $\boldsymbol{z} \in \mathsf{F}_1$ with the property that $\boldsymbol{x} \in (\boldsymbol{y}, \boldsymbol{z})$. But $\boldsymbol{x} \in \mathsf{F}_2$ is a convex combination of $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{C}$. Therefore, $\boldsymbol{y}, \boldsymbol{z} \in \mathsf{F}_2$ by the definition of the face $\mathsf{F}_2$. This contradicts the requirement that $\boldsymbol{y} \in \mathsf{F}_1 \setminus \mathsf{F}_2$.

For the second statement, choose a point $\boldsymbol{x} \in \mathsf{C}$. Define the closed convex set

$$\mathsf{F}_{\boldsymbol{x}} := \bigcap \{\mathsf{F} \lhd \mathsf{C} : \boldsymbol{x} \in \mathsf{F}\}.$$

Recall that $\lhd$ denotes the "face of" relation. Since the intersection of faces is a face, we see that $\mathsf{F}_{\boldsymbol{x}} \lhd \mathsf{C}$.

If $\boldsymbol{x} \in \operatorname{relint} \mathsf{F}_{\boldsymbol{x}}$, the first statement ensures that no other face of $\mathsf{C}$ contains $\boldsymbol{x}$ in its relative interior. Therefore, $\boldsymbol{x}$ is in the relative interior of a unique face.

On the other hand, imagine that $\boldsymbol{x} \in \operatorname{relbd} \mathsf{F}_{\boldsymbol{x}}$. We will see that this case is impossible. Indeed, the closed convex set $\mathsf{F}_{\boldsymbol{x}}$ has a proper supporting hyperplane $\mathsf{H}$ at $\boldsymbol{x}$; that is, $\boldsymbol{x} \in \mathsf{H}$, but $\mathsf{F}_{\boldsymbol{x}} \cap \mathsf{H} \subsetneqq \mathsf{F}_{\boldsymbol{x}}$. By construction, $\mathsf{F}_{\boldsymbol{x}} \cap \mathsf{H}$ is an exposed face of $\mathsf{F}_{\boldsymbol{x}}$, so it is a face of $\mathsf{F}_{\boldsymbol{x}}$. By transitivity of faces, $(\mathsf{F}_{\boldsymbol{x}} \cap \mathsf{H}) \lhd \mathsf{C}$. But $\boldsymbol{x} \in (\mathsf{F}_{\boldsymbol{x}} \cap \mathsf{H}) \subsetneqq \mathsf{F}_{\boldsymbol{x}}$. This contradicts the minimality of the face $\mathsf{F}_{\boldsymbol{x}}$. □

## 6.3 Normal Cones

Next, we introduce the concept of a normal cone of a point of a convex set. The normal cone collects all of the outer normals to hyperplanes that weakly separate the point from the set.

**Definition 6.3.1** (Normal cone). Let $\mathsf{C} \subset \mathbb{R}^d$ be a closed convex set. For a point $\boldsymbol{x} \in \mathsf{C}$, define the *normal cone* to the set $\mathsf{C}$ at the point $\boldsymbol{x}$ as

$$\begin{aligned} \mathsf{N}_{\mathsf{C}}(\boldsymbol{x}) &:= \{\boldsymbol{s} \in \mathbb{R}^d : \langle \boldsymbol{s},\, \boldsymbol{x} \rangle = h_{\mathsf{C}}(\boldsymbol{s})\} \\ &= \left\{\boldsymbol{s} \in \mathbb{R}^d : \langle \boldsymbol{s},\, \boldsymbol{x} \rangle = \max\nolimits_{\boldsymbol{y} \in \mathsf{C}} \langle \boldsymbol{s},\, \boldsymbol{y} \rangle \right\}. \end{aligned}$$

As usual, $h_{\mathsf{C}}(\boldsymbol{s})$ denotes the value of the support function of $\mathsf{C}$ in direction $\boldsymbol{s} \in \mathbb{R}^d$.

First, let us confirm that normal cones are indeed cones.

**Proposition 6.3.2** (The normal cone is a cone). *The normal cone $\mathsf{N}_{\mathsf{C}}(\boldsymbol{x})$ is a closed convex cone.*

*Proof.* Since $\boldsymbol{x} \in \mathsf{C}$, a direction $\boldsymbol{s}$ belongs to $\mathsf{N}_{\mathsf{C}}(\boldsymbol{x})$ if and only if

$$\langle \boldsymbol{s},\, \boldsymbol{y} - \boldsymbol{x} \rangle \le 0 \quad \text{for all } \boldsymbol{y} \in \mathsf{C}.$$

In other words, the normal cone is an intersection of closed halfspaces that contain the origin. Thus, it is a closed convex cone. □

Next, we develop an alternative representation of the normal cone in terms of the metric projector onto the convex set.

**Proposition 6.3.3** (Normal cones and projections). *Let $\mathsf{C}$ be a closed convex set, and let $\boldsymbol{x} \in \mathsf{C}$. The normal cone can be written as*

$$\mathsf{N}_{\mathsf{C}}(\boldsymbol{x}) = \{\boldsymbol{y} - \boldsymbol{x} : \operatorname{proj}_{\mathsf{C}}(\boldsymbol{y}) = \boldsymbol{x}\}.$$

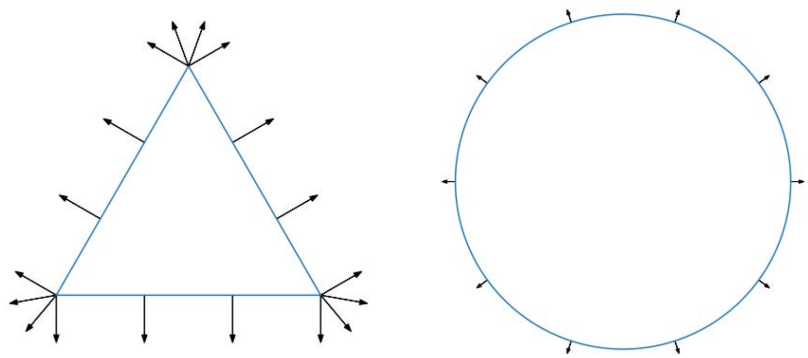

**Figure 6.1** (Normal cones). This diagram describes the normal cones of a triangle and a disk. For a point on the boundary of each object, the normal cone contains all of the arrows that emanate from that point.

The set $\boldsymbol{x} + \mathsf{N}_{\mathsf{C}}(\boldsymbol{x})$ is sometimes called the *normal bundle* of $\mathsf{C}$ at $\boldsymbol{x}$. The normal bundle contains all of the points in space whose metric projection onto $\mathsf{C}$ is $\boldsymbol{x}$.

*Proof.* Suppose that $\boldsymbol{s} \in \mathsf{N}_{\mathsf{C}}(\boldsymbol{x})$. By definition,

$$\max_{\boldsymbol{z} \in \mathsf{C}} \langle \boldsymbol{s}, \boldsymbol{z} \rangle = \langle \boldsymbol{s}, \boldsymbol{x} \rangle.$$

Consider a point of the form $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{s}$. Then

$$\langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{z} \rangle \leq \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{x} \rangle \quad \text{for all } \boldsymbol{z} \in \mathsf{C}.$$

Equivalently,

$$\langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{z} - \boldsymbol{x} \rangle \leq 0 \quad \text{for all } \boldsymbol{z} \in \mathsf{C}.$$

Since $\boldsymbol{x} \in \mathsf{C}$, we see that $\boldsymbol{x}$ satisfies the variational formula for the projection $\operatorname{proj}_{\mathsf{C}}(\boldsymbol{y})$. The result follows. □

We started by defining the normal cone of the convex set $\mathsf{C}$ at a point $\boldsymbol{x}$. In fact, the normal cone is the same for every point in the relative interior of a face. It is natural, therefore, to parameterize normal cones in terms of faces, rather than points.

**Proposition 6.3.4** (The normal cone depends only the face)**.** *Assume that* $\mathsf{F}$ *is a face of a closed convex set* $\mathsf{C}$*. If* $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{relint} \mathsf{F}$*, then* $\mathsf{N}_{\mathsf{C}}(\boldsymbol{x}) = \mathsf{N}_{\mathsf{C}}(\boldsymbol{y})$*.*

*Proof.* We prove this proposition by contradiction. Suppose $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{relint} \mathsf{F}$, but their normal cones are different.

Interchanging $\boldsymbol{x}$ and $\boldsymbol{y}$ if necessary, we can find a direction $\boldsymbol{s} \in \mathbb{R}^d$ such that $\boldsymbol{s} \in \mathsf{N}_{\mathsf{C}}(\boldsymbol{x}) \setminus \mathsf{N}_{\mathsf{C}}(\boldsymbol{y})$. Since $\boldsymbol{s} \notin \mathsf{N}_{\mathsf{C}}(\boldsymbol{y})$ but $\boldsymbol{s} \in \mathsf{N}_{\mathsf{C}}(\boldsymbol{x})$, the definition of the normal cone implies that

$$\langle \boldsymbol{s}, \boldsymbol{y} \rangle < \max_{\boldsymbol{z} \in \mathsf{C}} \langle \boldsymbol{s}, \boldsymbol{z} \rangle = \max_{\boldsymbol{z} \in \mathsf{F}} \langle \boldsymbol{s}, \boldsymbol{z} \rangle = \langle \boldsymbol{s}, \boldsymbol{x} \rangle =: \alpha.$$

We determine that $\boldsymbol{x} \in \mathsf{F} \cap \mathsf{H}_{\boldsymbol{s},\alpha}$ but $\boldsymbol{y} \notin \mathsf{F} \cap \mathsf{H}_{\boldsymbol{s},\alpha}$. In other words, $\boldsymbol{x}$ belongs to a proper face of $\mathsf{F}$, so $\boldsymbol{x} \in \operatorname{relbd} \mathsf{F}$. This is a contradiction. □

As a consequence of Proposition 6.3.4, we have a well-defined concept of the normal cone of a face of a convex set.

**Definition 6.3.5** (Normal cone of a face)**.** If $\mathsf{F}$ is a face of a closed convex set $\mathsf{C}$, the *normal cone of the face* is defined as

$$\mathsf{N}_{\mathsf{C}}(\mathsf{F}) := \mathsf{N}_{\mathsf{C}}(\boldsymbol{x}) \quad \text{for a point } \boldsymbol{x} \in \operatorname{relint} \mathsf{F}.$$

Proposition 6.3.4 ensures that the specific choice of $\boldsymbol{x}$ is immaterial.

## 6.4 The Tiling Induced by a Convex Set

In this section, we develop a fundamental decomposition of Euclidean space induced by a closed convex set. The normal bundles of the points in the set are disjoint, and they exhaust all of the space. This construction can be simplified somewhat because each point in a face has the same normal cone.

**Theorem 6.4.1** (Tiling induced by a convex set)**.** *Suppose $\mathsf{C} \subset \mathbb{R}^d$ is closed and convex. Then we have the decomposition*

$$\mathbb{R}^d = \dot{\bigcup}_{\mathsf{F} \lhd \mathsf{C}} ((\operatorname{relint} \mathsf{F}) + \mathsf{N}_{\mathsf{C}}(\mathsf{F})).$$

*The dot indicates that the union is disjoint.*

*Proof.* Since the projector $\operatorname{proj}_{\mathsf{C}} : \mathbb{R}^d \to \mathsf{C}$ is a surjection, every point in $\mathbb{R}^d$ lies in the preimage of the metric projector. Therefore,

$$\begin{aligned}
\mathbb{R}^d &= \dot{\bigcup}_{\mathsf{F} \lhd \mathsf{C}} \operatorname{proj}_{\mathsf{C}}^{(-1)}(\operatorname{relint} \mathsf{F}) \\
&= \dot{\bigcup}_{\mathsf{F} \lhd \mathsf{C}} \dot{\bigcup}_{\boldsymbol{x} \in \operatorname{relint} \mathsf{F}} \operatorname{proj}_{\mathsf{C}}^{(-1)}(\boldsymbol{x}) \\
&= \dot{\bigcup}_{\mathsf{F} \lhd \mathsf{C}} \dot{\bigcup}_{\boldsymbol{x} \in \operatorname{relint} \mathsf{F}} (\boldsymbol{x} + \mathsf{N}_{\mathsf{C}}(\boldsymbol{x})) \\
&= \dot{\bigcup}_{\mathsf{F} \lhd \mathsf{C}} \dot{\bigcup}_{\boldsymbol{x} \in \operatorname{relint} \mathsf{F}} (\boldsymbol{x} + \mathsf{N}_{\mathsf{C}}(\mathsf{F})) \\
&= \dot{\bigcup}_{\mathsf{F} \lhd \mathsf{C}} ((\operatorname{relint} \mathsf{F}) + \mathsf{N}_{\mathsf{C}}(\mathsf{F})).
\end{aligned}$$

The first relation follows from Proposition 6.2.2, which ensures that each point $\boldsymbol{x} \in \mathsf{C}$ is contained in the relative interior of a unique face. Moreover, since the relative interiors of the faces are disjoint, we have decomposed $\mathbb{R}^d$ as a disjoint union. To reach the second line, we break up $\operatorname{relint} \mathsf{F}$ into its constituent points; each has a distinct preimage because $\operatorname{proj}_{\mathsf{C}}$ is a function. The third line depends on Proposition 6.3.3, which relates the metric projector to the normal bundle. Finally, we use Proposition 6.3.4 to see that the the normal cone of a point $\boldsymbol{x}$ depends only on the face in which it appears. □

## 6.5 Proper Faces of a Polytope are Exposed

We now specialize our attention to polytopes. The facial structure of a polytope is simpler than the facial structure of a general convex set. As a consequence, the tiling described in Theorem 6.4.1 enjoys additional properties.

The goal of this section is to prove that every face of a polytope is exposed. First, we show that an exposed face of a polytope is the convex hull of the vertices contained in the face.

**Proposition 6.5.1** (Exposed faces of a polytope). *Let* $\mathsf{P} = \operatorname{conv}\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$ *be a polytope, and let* $\mathsf{F}$ *be an exposed face of* $\mathsf{P}$*. Then* $\mathsf{F} = \operatorname{conv}\{\boldsymbol{x}_i : \boldsymbol{x}_i \in \mathsf{F}\}$*.*

*Proof.* Define the set $I = \{i : \boldsymbol{x}_i \in \mathsf{F}\} \subset \{1, \dots, n\}$ of indices of the vertices of $\mathsf{P}$ that are contained in the face $\mathsf{F}$. Since the face is convex, $\operatorname{conv}\{\boldsymbol{x}_i : i \in I\} \subset \mathsf{F}$. Our task is to establish the reverse inclusion.

Since $\mathsf{F}$ is an exposed face, then there exists a linear functional $\varphi$ that exposes the face. That is, $\varphi(\boldsymbol{z}) \leq 1$ for all $\boldsymbol{z} \in \mathsf{P}$, and $\varphi(\boldsymbol{z}) = 1$ if and only if $\boldsymbol{z} \in \mathsf{F}$.

Now, select a point $\boldsymbol{x} \in \mathsf{F}$. Since $\mathsf{F} \subset \mathsf{P}$, we can write the point $\boldsymbol{x}$ as a convex combination of the vertices of $\mathsf{P}$:

$$\boldsymbol{x} = \sum\nolimits_{i=1}^{n} \lambda_i \boldsymbol{x}_i \quad \text{where } \boldsymbol{\lambda} \in \Delta_n.$$

If there exists $j \notin I$ with $\lambda_j > 0$, then

$$\begin{aligned}\varphi(\boldsymbol{x}) &= \sum\nolimits_{i=1}^{n} \lambda_i \varphi(\boldsymbol{x}_i) = \lambda_j \varphi(\boldsymbol{x}_j) + \sum\nolimits_{i \neq j} \lambda_i \varphi(\boldsymbol{x}_i) \\ &\leq \lambda_j \varphi(\boldsymbol{x}_j) + \sum\nolimits_{i \neq j} \lambda_i < \lambda_j + \sum\nolimits_{i \neq j} \lambda_i = 1.\end{aligned}$$

However, this contradicts the assumption that $\boldsymbol{x} \in \mathsf{F}$.

In short, every point in the face $\mathsf{F}$ can be expressed as a convex combination of the vertices $\{\boldsymbol{x}_i : i \in I\}$. Thus, $\mathsf{F} \subset \operatorname{conv}\{\boldsymbol{x}_i : i \in I\}$. □

Proposition 6.5.1 has some interesting implications for the structure of a polytope.

**Corollary 6.5.2** (Properties of polytope faces). *The exposed faces of a polytope enjoy the following properties.*

1. *The exposed faces are polytopes.*
2. *The number of exposed faces is finite.*

*Proof.* First, since an exposed face of the polytope is the convex hull of a finite point set, it is a polytope.

Second, suppose that the polytope has $n$ vertices. Every exposed face is the convex hull of a subset of these $n$ vertices. Therefore, the number of exposed faces cannot exceed $2^n$. □

In contrast with the general case, exposed faces of a polytope are transitive.

**Corollary 6.5.3** (Transitivity of exposed faces of a polytope). *For a polytope, an exposed face of an exposed face is an exposed face.*

*Proof.* Consider a polytope $\mathsf{P} = \operatorname{conv}\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\}$. Let $\mathsf{G}$ be an exposed face of $\mathsf{P}$, and let $\mathsf{F}$ be an exposed face of $\mathsf{G}$. According to Proposition 6.5, we may assume that

$$\mathsf{F} = \operatorname{conv}\{\boldsymbol{x}_i : i \in I\} \quad \text{and} \quad \mathsf{G} = \operatorname{conv}\{\boldsymbol{x}_j : j \in J\} \quad \text{where} \quad I \subset J.$$

By exposure, there exist linear functionals $\varphi, \psi$ and numbers $\alpha, \beta$ such that

$$\begin{aligned}\varphi(\boldsymbol{x}_i) &= \alpha \quad \text{for } i \in I; \qquad & \psi(\boldsymbol{x}_j) &= \beta \quad \text{for } j \in J; \\ \varphi(\boldsymbol{x}_i) &< \alpha \quad \text{for } i \in J \setminus I; \qquad & \psi(\boldsymbol{x}_j) &< \beta \quad \text{for } j \notin J.\end{aligned}$$

For sufficiently small $\varepsilon$, the linear functional $\xi = \varphi + \varepsilon\psi$ has the property that

$$\begin{aligned} \xi(\boldsymbol{x}_i) &= \alpha + \varepsilon\beta \quad \text{for } i \in I; \\ \xi(\boldsymbol{x}_i) &< \alpha + \varepsilon\beta \quad \text{for } i \notin I. \end{aligned}$$

This is a simple consequence of the fact that $\mathsf{P}$ has a finite number of vertices. It follows that $\xi$ weakly separates $\mathsf{F}$ from $\mathsf{P}$, and $\mathsf{F}$ is an exposed face of $\mathsf{P}$. □

We arrive at the main result of the section.

**Theorem 6.5.4** (Proper faces of a polytope are exposed)**.** *Every proper face of a polytope is an exposed face.*

*Proof.* We prove this result by induction over the dimension of the polytope $\mathsf{P}$. If $\dim \mathsf{P} = 0$, the statement is vacuous because the polytope has no proper face. Suppose that $\dim \mathsf{P} = d$, and we have established the result for polytopes of lower dimension.

First, we claim that each proper face $\mathsf{F}$ of $\mathsf{P}$ is contained in an exposed face $\mathsf{F}'$ of $\mathsf{P}$. Proposition 6.2.2 implies that $\operatorname{relint} \mathsf{F} \cap \operatorname{relint} \mathsf{P} = \emptyset$. Thus, we can find a hyperplane $\mathsf{H}$ that weakly separates $\mathsf{F}$ from $\mathsf{P}$. Define $\mathsf{F}' = \mathsf{P} \cap \mathsf{H}$. Since $\mathsf{F} \subset \mathsf{P}$, we see that $\mathsf{F}'$ is an exposed face of $\mathsf{P}$ that contains $\mathsf{F}$.

If $\mathsf{F} = \mathsf{F}'$, it follows that $\mathsf{F}$ is an exposed face of $\mathsf{P}$. Suppose instead that $\mathsf{F}$ is a proper face of $\mathsf{F}'$. Then we must have $\dim \mathsf{F} \leq d - 1$. The exposed face $\mathsf{F}'$ is a polytope because of Corollary 6.5.2(1). The induction hypothesis now implies that $\mathsf{F}$ is an exposed face of $\mathsf{F}'$. Corollary 6.5.3 ensures that $\mathsf{F}$ is also an exposed face of $\mathsf{P}$. □

## 6.6 Normal Cones of Polyhedra

Next, we develop a structural result on the normal cones of a polyhedron. This result shows that the normal cone at a point is generated by the active constraints. It also allows us to compute the dimension of the normal cone.

**Proposition 6.6.1** (Normal cones of a polyhedron)**.** *Let $\mathsf{P} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s}_i,\, \boldsymbol{x} \rangle \leq \alpha_i, i = 1, \dots, m\}$ be a polyhedron. For a point $\boldsymbol{x} \in \mathsf{P}$, define the set $I(\boldsymbol{x}) = \{i : \langle \boldsymbol{s}_i,\, \boldsymbol{x} \rangle = \alpha_i\}$ of active constraints. Then $\mathsf{N}_{\mathsf{P}}(\boldsymbol{x}) = \operatorname{cone}\{\boldsymbol{s}_i : i \in I(\boldsymbol{x})\}$.*

*Proof.* Without loss of generality, we can assume that $\boldsymbol{x} = \mathbf{0}$ by considering the translated polyhedron $\mathsf{P} - \boldsymbol{x}$. Define another polyhedron that is generated only by the constraints that are active at the origin:

$$\mathsf{C} := \{\boldsymbol{z} \in \mathbb{R}^d : \langle \boldsymbol{s}_i,\, \boldsymbol{z} \rangle \leq 0 : i \in I(\mathbf{0})\}.$$

The normal cone of a polyhedron is locally determined, so $\mathsf{N}_{\mathsf{C}}(\mathbf{0})$ is the same as $\mathsf{N}_{\mathsf{P}}(\mathbf{0})$.

Define the conic hull of the active constraints:

$$\mathsf{K} = \operatorname{conv}\{\boldsymbol{s}_i : i \in I(\mathbf{0})\}.$$

By convexity of normal cones, $\mathsf{K} \subset \mathsf{N}_{\mathsf{C}}(\mathbf{0}) = \mathsf{N}_{\mathsf{P}}(\mathbf{0})$. It remains to develop the reverse inclusion.

We prove that $\mathsf{N_C}(\mathbf{0}) \subset \mathsf{K}$ by contradiction. Suppose that there exists a direction $\boldsymbol{v} \in \mathsf{N_C}(\mathbf{0}) \backslash \mathsf{K}$. Since $\boldsymbol{v} \in \mathsf{N_C}(\mathbf{0})$,

$$\langle \boldsymbol{v}, \boldsymbol{x} \rangle \leq 0 \quad \text{for all } \boldsymbol{x} \in \mathsf{C}.$$

Because $\boldsymbol{v} \notin \mathsf{K}$, we can separate the point $\boldsymbol{v}$ from the closed convex set $\mathsf{K}$. Indeed, there exists $\boldsymbol{z} \in \mathbb{R}^d$ such that

$$\langle \boldsymbol{v}, \boldsymbol{z} \rangle > 0 = \max_{\boldsymbol{s} \in \mathsf{K}} \langle \boldsymbol{s}, \boldsymbol{z} \rangle.$$

However, by construction of $\mathsf{K}$,

$$\max_{i \in I(\mathbf{0})} \langle \boldsymbol{s}_i, \boldsymbol{z} \rangle = \max_{\boldsymbol{s} \in \mathsf{K}} \langle \boldsymbol{s}, \boldsymbol{z} \rangle = 0.$$

As a consequence, we determine that $\boldsymbol{z} \in \mathsf{C}$. The separation condition requires that $\langle \boldsymbol{v}, \boldsymbol{z} \rangle > 0$. This outcome contradicts the relation $\langle \boldsymbol{v}, \boldsymbol{z} \rangle \leq 0$, which holds because $\boldsymbol{z} \in \mathsf{C}$. □

We have the following important corollary, which states that the dimension of a face and its normal cone are complementary.

**Corollary 6.6.2** (Dimension of the normal cone). *Let* $\mathsf{F}$ *be a face of a polyhedron* $\mathsf{P} \subset \mathbb{R}^d$. *Then*

$$\dim \mathsf{F} + \dim \mathsf{N_P}(\mathsf{F}) = d.$$

*Proof.* Consider a polyhedron

$$\mathsf{P} = \{ \boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s}_i, \boldsymbol{x} \rangle \leq \alpha_i \text{ for } i = 1, \dots, m \}.$$

Let $\mathsf{F}$ be a face of $\mathsf{P}$, and let $\boldsymbol{x} \in \operatorname{relint} \mathsf{F}$. Then the face can be written in terms of the active constraints $I(\boldsymbol{x})$:

$$\mathsf{F} = \{ \boldsymbol{x} \in \mathsf{P} : \langle \boldsymbol{s}_i, \boldsymbol{x} \rangle = \alpha_i \text{ for } i \in I(\boldsymbol{x}) \}.$$

(Justify this claim!) As a consequence, the dimension of the face is

$$\dim \mathsf{F} = \operatorname{codim} \operatorname{lin} \{ \boldsymbol{s}_i : i \in I(\boldsymbol{x}) \}.$$

Proposition 6.6.1 shows that the dimension of the normal cone is

$$\dim \mathsf{N_F}(\boldsymbol{x}) = \dim \operatorname{lin} \{ \boldsymbol{s}_i : i \in I(\boldsymbol{x}) \}.$$

Together, these relations imply the statement. □

# Lecture 7: Hausdorff Distance

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*ACM 204, Fall 2018*
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*23 October 2018*

## 7.1 Agenda for Lecture 7

In this lecture, we explain how to measure the distance between two compact convex sets using the Hausdorff metric. This metric allows us to approximate any convex set by a sequence of polytopes. Next, we establish an important topological result, called the Blaschke selection theorem, which states that the convex bodies form a locally compact metric space. We conclude the lecture with two examples of Hausdorff continuous functions: the metric projection of a fixed point onto a convex body and the distance from a fixed point to a convex body.

1. Hausdorff distance
2. Approximation by polytopes
3. The Blaschke selection theorem
4. Continuity of the metric projection

## 7.2 Hausdorff Distance

We wish to address the following questions:

1. How do we measure the "distance" between two compact convex sets?
2. Can we approximate a compact convex set by a "simpler" one, e.g., a polytope?
3. What functions are preserved under approximation?

We will make these questions more precise and well-defined as our discussion progresses. Let us begin with the following definitions.

**Definition 7.2.1** (Convex body)**.** A *convex body* is a compact convex set, possibly empty. Introduce the class $\mathscr{C}_d$ of all convex bodies in $\mathbb{R}^d$.

**Definition 7.2.2** (Hausdorff distance)**.** Let $\mathsf{C}, \mathsf{K}$ be convex bodies in $\mathscr{C}_d$. Then we define the *Hausdorff distance*

$$\mathrm{dist}_{\mathrm{H}}(\mathsf{C}, \mathsf{K}) := \inf\{\varepsilon > 0 : \mathsf{C} \subset \mathsf{K} + \varepsilon \mathsf{B}_d \text{ and } \mathsf{K} \subset \mathsf{C} + \varepsilon \mathsf{B}_d\}.$$

It is important that the we define the Hausdorff distance to be symmetrical. Conceptually, the Hausdorff distance tells us how much we need to enlarge one of the bodies in order to enclose the other, and vice versa. More precisely, what is the least $\varepsilon$ such that every point in $\mathsf{C}$ is within $\varepsilon$ distance of some point in $\mathsf{K}$ and every point in $\mathsf{K}$ is within $\varepsilon$ distance of some point in $\mathsf{C}$?

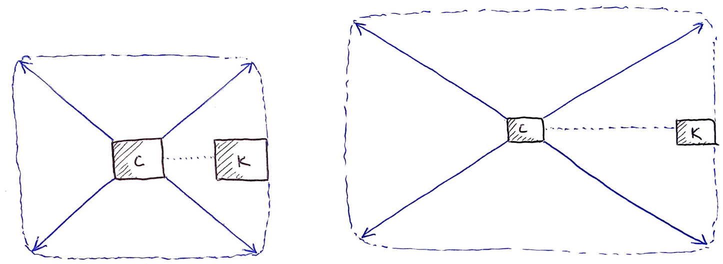


**Figure 7.1** (Hausdorff distance illustrations). Note that the Hausdorff distance depends on the position of the convex bodies relative to each other. [left] If the bodies are close to each other then the ball radius $\varepsilon$ for the set $\mathsf{C} + \varepsilon \mathsf{B}_d$ does not have to be too large in order to enclose $\mathsf{K}$ (and vice versa). [right] If the bodies are further apart, then $\varepsilon$ must be larger.

An equivalent formulation of the definition is as follows:

$$\begin{aligned} \operatorname{dist}_{\mathrm{H}}(\mathsf{C}, \mathsf{K}) &= \max\left\{ \max_{\boldsymbol{x} \in \mathsf{C}} \operatorname{dist}(\boldsymbol{x}, \mathsf{K});\ \max_{\boldsymbol{y} \in \mathsf{K}} \operatorname{dist}(\boldsymbol{y}, \mathsf{C}) \right\} \\ &= \max\left\{ \max_{\boldsymbol{x} \in \mathsf{C}} \min_{\boldsymbol{y} \in \mathsf{K}} \|\boldsymbol{x} - \boldsymbol{y}\|;\ \max_{\boldsymbol{y} \in \mathsf{K}} \min_{\boldsymbol{x} \in \mathsf{C}} \|\boldsymbol{x} - \boldsymbol{y}\| \right\}. \end{aligned}$$

We can use minimum and maximum operations instead of infimum and supremum because we are dealing with compact sets, which implies that they are closed.

## 7.3 Approximation by Polytopes

The Hausdorff distance is a metric (see Theorem 7.6.1 for proof). Thus, we can equip $\mathscr{C}_d$ with $\operatorname{dist}_{\mathrm{H}}$ to form a metric space. This distance metric gives us a notion of approximation; we consider three instances in Figure 7.2.

**Theorem 7.3.1** (Approximation by polytopes)**.** *Let* $\mathsf{C}$ *be a convex body with dimension* $d$*, and let* $\varepsilon > 0$*. Then there exists a polytope* $\mathsf{P}_\varepsilon \subset \mathsf{C}$ *such that* $\operatorname{dist}_{\mathrm{H}}(\mathsf{P}_\varepsilon, \mathsf{C}) \le \varepsilon$*.*

In other words, for any positive distance $\varepsilon$, we can produce a polytope that is a "good" approximation of the set, where the criterion of how "good" the approximation is determined by the Hausdorff distance to $\mathsf{C}$.)

*Proof.* Consider the open cover

$$\mathsf{C} \subset \bigcup\nolimits_{\boldsymbol{x} \in \mathsf{C}} \mathsf{N}(\boldsymbol{x}; \varepsilon),$$

where $\mathsf{N}(\boldsymbol{x}; \varepsilon)$ denotes the open ball (neighborhood) of radius $\varepsilon$ centered at $\boldsymbol{x}$. The convex body $\mathsf{C}$ is compact, so there exists a finite subcover:

$$\mathsf{C} \subset \bigcup\nolimits_{\boldsymbol{x} \in \mathsf{S}} \mathsf{N}(\boldsymbol{x}; \varepsilon), \quad \text{where } \mathsf{S} \subset \mathsf{C} \text{ and } \#\mathsf{S} < \infty.$$

Define the polytope $\mathsf{P}_\varepsilon$ to be the convex hull of the finite set $\mathsf{S}$.

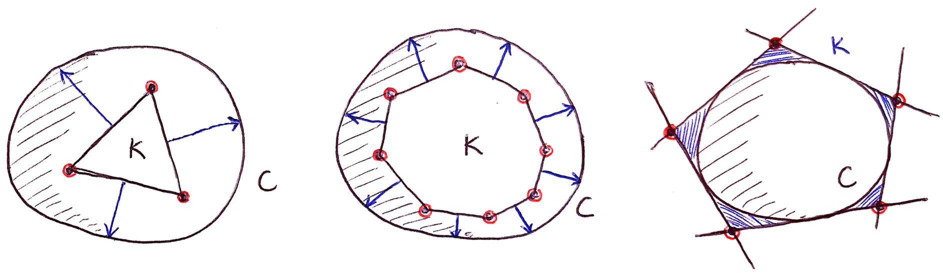


**Figure 7.2** (Approximation by polytopes). Consider approximating the convex body C using the polytope K. [left] We choose three points in C (red) and take the approximating polytope K to be their convex hull. [center] We can obtain another approximation by taking the convex hull of more points. This approximation is better than the leftmost approximation because we do not need to expand K as much in order to enclose C; that is, $\mathrm{dist}_{\mathrm{H}}(\mathsf{C}, \mathsf{K})$ is larger in the leftmost case than in the middle case. [right] We can also approximate by taking the polytope K to be the convex hull of points selected outside the set C. The set K is the convex hull of the points of intersection of five supporting hyperplanes to C.

We will verify that $\mathsf{P}_\varepsilon$ and C lie within Hausdorff distance $\varepsilon$. Since S is a subset of C and because C is convex, the convex hull of S is also a subset of C. Hence, $\mathsf{P}_\varepsilon \subset \mathsf{C}$. On the other hand, the finite subcover of C is a subset of $\mathsf{P}_\varepsilon + \varepsilon \mathsf{B}_d$. That is,

$$\mathsf{C} \subset \bigcup_{\boldsymbol{x} \in \mathsf{S}} \mathsf{N}(\boldsymbol{x}; \varepsilon) \subset \mathsf{P}_\varepsilon + \varepsilon \mathsf{B}_d.$$

Thus, we conclude $\mathrm{dist}_{\mathrm{H}}(\mathsf{P}_\varepsilon, \mathsf{C}) \leq \varepsilon$. □

## 7.4 The Blaschke Selection Theorem

We can define a notion of convergence of sets with respect to the Hausdorff distance, which we denote using the usual arrow notation ($\to$). For instance, the previous theorem can be expressed as follows. There is a sequence $\{\mathsf{P}_n : n \in \mathbb{N}\}$ of polytopes in $\mathscr{C}_d$ with the property that $\mathsf{P}_n \to \mathsf{C}$ as $n \to \infty$.

Another topological concept is boundedness with respect to the metric.

**Definition 7.4.1** (Boundedness). A sequence $\{\mathsf{C}_i : i \in \mathbb{N}\}$ is *bounded* if each set $\mathsf{C}_i$ is contained in a closed Euclidean ball $\bar{\mathsf{N}}(\mathbf{0}; \varrho)$ with radius $\varrho$, centered at the origin.

We now turn to the fundamental result about the topology of the metric space of convex bodies equipped with the Hausdorff distance.

**Theorem 7.4.2** (Blaschke selection). *The metric space $(\mathscr{C}_d, \mathrm{dist}_{\mathrm{H}})$ is locally compact. In other words, a bounded sequence of convex bodies in $\mathscr{C}_d$ contains a convergent subsequence.*

It may be useful to state the result in more mathematical detail. Consider a sequence $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3, \cdots$ of nonempty convex bodies in $\mathscr{C}_d$, and assume that each $\mathsf{C}_i \subset \varrho \mathsf{B}_d$ for some $\varrho > 0$. Then there is a subsequence $\{\mathsf{C}_{i_j} : j \in \mathbb{N}\}$ with the property that $\mathsf{C}_{i_j} \to \mathsf{C}$, where the limit C is a nonempty convex body in $\mathscr{C}_d$.

Before we give a proof of the theorem, let us consider some simple applications.

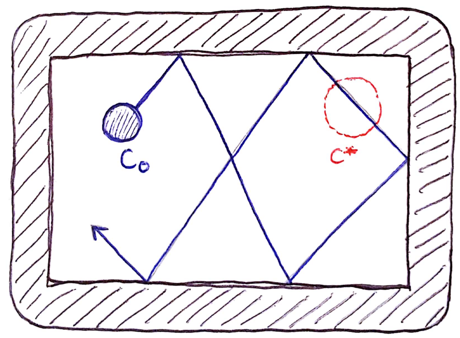


**Figure 7.3** (Positions of a billiard ball). We consider images $\mathsf{C}_i := \boldsymbol{Q}_i\mathsf{C} + \boldsymbol{x}_i$ of a billiard ball at times $i = 1, 2, 3, \cdots$. Then $\{\mathsf{C}_i\}$ has a subsequence $\{\mathsf{C}_{i_j} : j \in \mathbb{N}\}$ that converges to a convex body $\mathsf{C}_\star$. In other words, there is a location where the ball passes arbitrarily close an infinite number of times.

**Example 7.4.3** (Approximation by polytopes). Consider the family of polytopes $\{\mathsf{P}_\varepsilon : \varepsilon > 0\}$ that we constructed in Theorem 7.3.1 to approximate the convex body $\mathsf{C}$. Then we can extract a convergent sequence, say $\{\mathsf{P}_{\varepsilon_i} : i \in \mathbb{N}\}$, with limit $\mathsf{C}_\star$. In fact, the limit $\mathsf{C}_\star = \mathsf{C}$ because the polytopes approximate $\mathsf{C}$ arbitrarily well, and a convergent sequence in a metric space has a unique limit.

**Example 7.4.4** (Poincaré recurrence). Here is an amusing example related to the Poincaré recurrence theorem. Fix a convex body $\mathsf{C} \in \mathscr{C}_d$. Let $\boldsymbol{x}_i$ be arbitrary points in a compact set $\Omega$, and let $\boldsymbol{Q}_i \in \mathbb{R}^{d\times d}$ be arbitrary rotations. We can define a sequence of convex bodies via

$$\mathsf{C}_i := \boldsymbol{Q}_i\mathsf{C} + \boldsymbol{x}_i \quad \text{for } i \in \mathbb{N}.$$

Think about the positions of a billiard ball at discrete time instants as the ball cascades around the table. By the Blaschke selection theorem, the sequence $\{\mathsf{C}_i\}$ admits a convergent subsequence, with limit $\mathsf{C}_\star$. For any $\varepsilon > 0$, there are an infinite number of times when $\operatorname{dist}_{\mathrm{H}}(\mathsf{C}_i, \mathsf{C}_\star) \leq \varepsilon$. In a sense, the limiting body $\mathsf{C}_\star$ is a frequent location of the moving set $\mathsf{C}_i$. See Figure 7.3.

Let us go ahead and establish the result. Further discussion appears in [Gru07, Sec. 6].

*Proof of the Blaschke selection theorem.* For simplicity of notation, denote $\mathsf{B} := \varrho\mathsf{B}_d$, the closed Euclidean ball centered at the origin and with radius $\varrho$.

We will invoke the Arzelà–Ascoli theorem: Consider a family of real-valued, continuous functions defined on a compact metric space. The family is compact if and only if it is equibounded and equicontinuous.

We plan to apply the following special case. Consider a sequence of functions $f_i : \mathsf{B} \to \mathbb{R}$ with the following properties:

1. (Equibounded). The numbers $|f_i(\boldsymbol{x})|$ are bounded above by a constant for all $\boldsymbol{x} \in \mathsf{B}$ and all $i \in \mathbb{N}$.

2. (1-Lipshitz). We have $|f_i(\boldsymbol{x}) - f_i(\boldsymbol{y})| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}$ and each $i \in \mathbb{N}$.

Then there exists a subsequence $\{f_{i_j}, j \in \mathbb{N}\}$ that converges uniformly to a limit $f_\star$. That is, for any $\varepsilon > 0$, there exists $J > 0$ such that for all $j \geq J$, we have $\|f_{i_j} - f_\star\|_{L_\infty(\mathsf{B})} < \varepsilon$.

For our scenario, define $f_i(\boldsymbol{x}) := \operatorname{dist}(\boldsymbol{x}; \mathsf{C}_i)$. Recall that $f_i$ is nonnegative, 1-Lipschitz, and convex. Moreover, it easy to check that $f_i$ is bounded above by the diameter of the Euclidean ball $\mathsf{B}$.

By Arzelà–Ascoli, there exists a subsequence $\{f_{i_j} : j \in \mathbb{N}\}$ that converges to a function $f_\star : \mathsf{B} \to \mathbb{R}$. It is immediate that the limit $f_\star$ is nonnegative and 1-Lipschitz. With some effort, it can be shown that the limit $f_\star$ is also convex.

Define $\mathsf{C}_\star$ to be the zero sublevel set of $f_\star$. That is, $\mathsf{C}_\star = \{\boldsymbol{x} \in \mathsf{B} : f_\star(\boldsymbol{x}) \leq 0\}$. The set $\mathsf{C}_\star$ is convex because $f_\star$ is convex. Furthermore, the set $\mathsf{C}_\star$ is closed because $f_\star$ is continuous. The set $\mathsf{C}_\star$ is obviously bounded because it is contained in the Euclidean ball $\mathsf{B}$. Hence $\mathsf{C}_\star$ is a convex body.

Finally, we need to show that there exists a subsequence in the family $\{\mathsf{C}_i\}$ of convex bodies that converges to $\mathsf{C}_\star$. That is, there exists $\{\mathsf{C}_{i_j} : j \in \mathbb{N}\}$ such that

$$\mathsf{C}_{i_j} \to \mathsf{C}_\star \quad \text{as } j \to \infty.$$

The subsequence of sets is precisely the one associated with the subsequence of functions:

$$\mathsf{C}_{i_j} = \{\boldsymbol{x} \in \mathsf{B} : f_{i_j}(\boldsymbol{x}) \leq 0\}.$$

See Theorem 7.6.2 for confirmation that the limit $\mathsf{C}_\star$ is nonempty.

Let $\varepsilon > 0$. Since $f_\star$ is continuous, there exists a $\delta > 0$ such that

$$\{\boldsymbol{x} \in \mathsf{B} : f_\star(\boldsymbol{x}) \leq \delta\} \subset \mathsf{C}_\star + \varepsilon \mathsf{B}_d.$$

(Recall that $\mathsf{B}_d$ is the unit Euclidean ball, not the dilated ball $\mathsf{B}$.) Moreover, because $f_{i_j} \to f_\star$ uniformly, we have $f_\star(\boldsymbol{x}) \leq f_{i_j}(\boldsymbol{x}) + \varepsilon$ uniformly for all $\boldsymbol{x} \in \mathsf{B}$ once the index $j$ is sufficiently large. As a consequence,

$$\mathsf{C}_{i_j} = \{\boldsymbol{x} \in \mathsf{B} : f_{i_j}(\boldsymbol{x}) \leq 0\} \subset \{\boldsymbol{x} \in \mathsf{B} : f_\star(\boldsymbol{x}) \leq \varepsilon\} = \mathsf{C}_\star + \varepsilon \mathsf{B}_d.$$

The same line of reasoning can be used to show that $\mathsf{C}_\star \subset \mathsf{C}_{i_j} + \varepsilon \mathsf{B}_d$ for all sufficiently large indices $j$. By definition of the Hausdorff distance, for all large $j \in \mathbb{N}$, we have $\operatorname{dist}_{\mathrm{H}}(\mathsf{C}_{i_j}, \mathsf{C}_\star) \leq \varepsilon$. Therefore, we conclude that $\mathsf{C}_{i_j} \to \mathsf{C}_\star$. □

## 7.5 Continuity of the Metric Projection

Continuous functions preserve convergence, so it is important to understand what functions on convex bodies are continuous with respect to the Hausdorff metric. As it happens, many natural functions on convex bodies are indeed continuous. The following theorem describes two of the most important examples.

**Theorem 7.5.1** (Metric projection is continuous). *Fix a point $\boldsymbol{x} \in \mathbb{R}^d$. Then $\operatorname{proj}(\boldsymbol{x}; \cdot)$ is a continuous function on $\mathscr{C}_d$. As a consequence, $\operatorname{dist}(\boldsymbol{x}; \cdot)$ is also continuous on $\mathscr{C}_d$.*

In more detail, suppose that $\mathsf{C}_i \to \mathsf{C}$ in the Hausdorff metric on $\mathscr{C}_d$. Then

$$\begin{aligned} \operatorname{proj}(\boldsymbol{x}; \mathsf{C}_i) &\to \operatorname{proj}(\boldsymbol{x}; \mathsf{C}); \\ \operatorname{dist}(\boldsymbol{x}; \mathsf{C}_i) &\to \operatorname{dist}(\boldsymbol{x}; \mathsf{C}_i). \end{aligned}$$

These relations will be very useful in our subsequent investigations.

*Proof.* Let $\mathsf{C}_i \to \mathsf{C}$ in the Hausdorff metric on $\mathscr{C}_d$. Define $\boldsymbol{y}_i := \operatorname{proj}(\boldsymbol{x}; \mathsf{C}_i)$ for each index $i \in \mathbb{N}$. We will make use of the following claims.

**Claim 7.5.2.** *There exists a subsequence* $\{\boldsymbol{y}_{i_j} : j \in \mathbb{N}\}$ *that converges to a limit* $\boldsymbol{y}_\star$.

*Proof.* Since $\mathsf{C}_i$ converges in Hausdorff metric, the entire sequence is enclosed within a fixed Euclidean ball. Therefore, the sequence $\{\boldsymbol{y}_i\}$ of metric projections must be bounded. A bounded sequence in Euclidean space admits a convergent subsequence. □

**Claim 7.5.3.** *The limit* $\boldsymbol{y}_\star$ *of the subsequence belongs to the limiting set* $\mathsf{C}$.

*Proof.* Recall that $\operatorname{dist}(\cdot; \mathsf{C})$ is a continuous function. Suppose that $\boldsymbol{y}_\star \notin \mathsf{C}$. Then there would exist an $\varepsilon > 0$ such that $\operatorname{dist}(\boldsymbol{y}_{i_j}; \mathsf{C}) > \varepsilon$ for an infinite sequence of indices $j$. But then $\operatorname{dist}_{\mathrm{H}}(\mathsf{C}_{i_j}, \mathsf{C}) \geq \operatorname{dist}(\boldsymbol{y}_{i_j}; \mathsf{C}) > \varepsilon > 0$. This inequality contradicts the fact that $\mathsf{C}_{i_j} \to \mathsf{C}$. □

**Claim 7.5.4.** *The point* $\boldsymbol{y}_\star = \operatorname{proj}(\boldsymbol{x}; \mathsf{C})$.

*Proof.* We will establish that $\boldsymbol{y}_\star$ meets the variational characterization of the metric projection of $\boldsymbol{x}$ onto $\mathsf{C}$. That is,

$$\langle \boldsymbol{x} - \boldsymbol{y}_\star,\, \boldsymbol{z} - \boldsymbol{y}_\star \rangle \leq 0 \quad \text{for all } \boldsymbol{z} \in \mathsf{C}.$$

This will follow as a consequence of the variational characterization of $\boldsymbol{y}_{i_j} = \operatorname{proj}(\boldsymbol{x}; \mathsf{C}_{i_j})$:

$$\langle \boldsymbol{x} - \boldsymbol{y}_{i_j},\, \boldsymbol{z} - \boldsymbol{y}_{i_j} \rangle \leq 0 \quad \text{for all } \boldsymbol{z} \in \mathsf{C}_{i_j}. \tag{7.5.1}$$

Note that this relation holds for each index $i_j$.

Fix a point $\boldsymbol{z} \in \mathsf{C}$. Because $\mathsf{C}_{i_j} \to \mathsf{C}$, we can construct a sequence $\{\boldsymbol{z}_{i_j}\}$ with $\boldsymbol{z}_{i_j} \in \mathsf{C}_{i_j}$ and $\boldsymbol{z}_{i_j} \to \boldsymbol{z}$. (We may need to extract a further subsequence to do so.) Adding and subtracting terms,

$$\begin{aligned}\langle \boldsymbol{x} - \boldsymbol{y}_\star,\, \boldsymbol{z} - \boldsymbol{y}_\star \rangle &= \langle (\boldsymbol{x} - \boldsymbol{y}_{i_j}) + (\boldsymbol{y}_{i_j} - \boldsymbol{y}_\star),\, (\boldsymbol{z} - \boldsymbol{z}_{i_j}) + (\boldsymbol{z}_{i_j} - \boldsymbol{y}_{i_j}) + (\boldsymbol{y}_{i_j} - \boldsymbol{y}_\star) \rangle \\ &= \langle \boldsymbol{x} - \boldsymbol{y}_{i_j},\, \boldsymbol{z}_{i_j} - \boldsymbol{y}_{i_j} \rangle + (\text{remainder terms}).\end{aligned}$$

By (7.5.1) and the fact that $\boldsymbol{z}_{i_j} \in \mathsf{C}_{i_j}$, the explicit term on the right-hand side is nonpositive.

Let us now carefully expand out the remainder terms:

$$\begin{aligned}(\text{remainder terms}) = {} & \langle \boldsymbol{x} - \boldsymbol{y}_{i_j},\, \boldsymbol{z} - \boldsymbol{z}_{i_j} \rangle + \langle \boldsymbol{x} - \boldsymbol{y}_{i_j},\, \boldsymbol{y}_{i_j} - \boldsymbol{y}_\star \rangle \\ & + \langle \boldsymbol{y}_{i_j} - \boldsymbol{y}_\star,\, \boldsymbol{z} - \boldsymbol{z}_{i_j} \rangle + \langle \boldsymbol{y}_{i_j} - \boldsymbol{y}_\star,\, \boldsymbol{z}_{i_j} - \boldsymbol{y}_{i_j} \rangle \\ & + \langle \boldsymbol{y}_{i_j} - \boldsymbol{y}_\star,\, \boldsymbol{y}_{i_j} - \boldsymbol{y}_\star \rangle.\end{aligned}$$

The first term converges to zero because $\{\boldsymbol{x} - \boldsymbol{y}_{i_j}\}$ is bounded and $\boldsymbol{z}_{i_j} \to \boldsymbol{z}$. The second, third, and fifth terms converge to zero because $\boldsymbol{y}_{i_j} \to \boldsymbol{y}_\star$. The fourth term converges to zero because $\{\boldsymbol{z}_{i_j} - \boldsymbol{y}_{i_j}\}$ is bounded. In short, the remainder converges to zero.

We can conclude that $\langle \boldsymbol{x} - \boldsymbol{y}_\star,\, \boldsymbol{z} - \boldsymbol{y}_\star \rangle \leq 0$ for every point $\boldsymbol{z} \in \mathsf{C}$. □

At this point, we have demonstrated that every cluster point of the sequence $\{\boldsymbol{y}_i\}$ coincides with $\operatorname{proj}(\boldsymbol{x}; \mathsf{C})$. But this implies that the entire sequence converges:

$$\boldsymbol{y}_i = \operatorname{proj}(\boldsymbol{x}; \mathsf{C}_i) \to \operatorname{proj}(\boldsymbol{x}; \mathsf{C}).$$

This establishes the first claim in the statement of the theorem.

Finally, we turn to the second claim. We have

$$\begin{aligned}\operatorname{dist}(\boldsymbol{x}; \mathsf{C}_i) &= \|\boldsymbol{x} - \operatorname{proj}(\boldsymbol{x}; \mathsf{C}_i)\| \\ &\to \|\boldsymbol{x} - \operatorname{proj}(\boldsymbol{x}; \mathsf{C})\| = \operatorname{dist}(\boldsymbol{x}; \mathsf{C}),\end{aligned}$$

where the second line follows directly from the first claim. □

Remark 7.5.5 (Hölder continuity of the projection). In fact, $\operatorname{proj}(\boldsymbol{x}, \cdot)$ is Hölder continuous. This follows from a more careful argument. See [Sch14, Lem. 1.8.11].

## 7.6 Supplementary Results

We close with some supplementary results about the Hausdorff distance.

### 7.6.1 Hausdorff Distance is a Metric

First, we verify that the Hausdorff distance satisfies all the properties required for it to be a metric.

**Theorem 7.6.1** (Hausdorff distance is a metric)**.** *The Hausdorff distance* $\operatorname{dist}_{\mathrm{H}}(\cdot, \cdot)$ *is a metric on the class* $\mathscr{C}_d$ *of convex bodies in* $\mathbb{R}^d$*.*

*Proof.* We establish the required properties one by one.

1. (Positive Definiteness). By definition of the Hausdorff distance, it is always nonnegative. Furthermore, if $\mathsf{C} = \mathsf{K}$, then $\operatorname{dist}_{\mathrm{H}}(\mathsf{C}, \mathsf{K}) = 0$. This is evident.

   Let us prove the converse. Suppose we have:

   $$\operatorname{dist}_{\mathrm{H}}(\mathsf{C}, \mathsf{K}) = \max\left\{\max_{\boldsymbol{x}\in\mathsf{C}} \operatorname{dist}(\boldsymbol{x}; \mathsf{K});\ \max_{\boldsymbol{y}\in\mathsf{K}} \operatorname{dist}(\boldsymbol{y}; \mathsf{C})\right\} = 0.$$

   This means that each individual argument is zero because $\operatorname{dist}(\cdot, \cdot)$ is a metric itself, and so it is nonnegative:

   $$\begin{aligned}\max_{\boldsymbol{x}\in\mathsf{C}} \operatorname{dist}(\boldsymbol{x}; \mathsf{K}) = 0 \quad &\text{implies} \quad \operatorname{dist}(\boldsymbol{x}, \mathsf{K}) = 0 \quad \text{for all } \boldsymbol{x} \in \mathsf{C}; \\ \max_{\boldsymbol{y}\in\mathsf{K}} \operatorname{dist}(\boldsymbol{x}; \mathsf{C}) = 0 \quad &\text{implies} \quad \operatorname{dist}(\boldsymbol{x}, \mathsf{C}) = 0 \quad \text{for all } \boldsymbol{y} \in \mathsf{K}.\end{aligned}$$

   By the definition of distance, this means that

   $$\min_{\boldsymbol{y}\in\mathsf{K}} \|\boldsymbol{x} - \boldsymbol{y}\| = 0 \quad \text{for all } \boldsymbol{x} \in \mathsf{C}; \tag{7.6.1}$$

   $$\min_{\boldsymbol{x}\in\mathsf{C}} \|\boldsymbol{x} - \boldsymbol{y}\| = 0 \quad \text{for all } \boldsymbol{y} \in \mathsf{K}. \tag{7.6.2}$$

   Relation (7.6.1) implies that, for every $\boldsymbol{x}$ in $\mathsf{C}$, we can find a corresponding $\boldsymbol{y}_\star \in \mathsf{K}$ at distance zero. In other words, $\boldsymbol{y}_\star = \boldsymbol{x} \in \mathsf{C}$. Hence $\mathsf{K} \subset \mathsf{C}$. Likewise, relation (7.6.2) implies that $\mathsf{C} \subset \mathsf{K}$. We determine that $\mathsf{C} = \mathsf{K}$.

   Combining with the previous result, $\operatorname{dist}_{\mathrm{H}}(\mathsf{C}, \mathsf{K}) = 0$ if and only if $\mathsf{C} = \mathsf{K}$.

2. (Symmetry). As stated before, the Hausdorff distance is defined to be symmetric:

$$\mathrm{dist}_{\mathrm{H}}(\mathsf{C},\mathsf{K}) = \max\left\{\max_{\boldsymbol{x}\in\mathsf{C}} \mathrm{dist}(\boldsymbol{x};\mathsf{K});\ \max_{\boldsymbol{y}\in\mathsf{K}} \mathrm{dist}(\boldsymbol{y};\mathsf{C})\right\} = \mathrm{dist}_{\mathrm{H}}(\mathsf{K},\mathsf{C}).$$

3. (Triangle Inequality). Let $\mathsf{C},\mathsf{K},\mathsf{S}$ be convex bodies in $\mathscr{C}_d$. We want to show that

$$\mathrm{dist}_{\mathrm{H}}(\mathsf{C},\mathsf{K}) \le \mathrm{dist}_{\mathrm{H}}(\mathsf{C},\mathsf{S}) + \mathrm{dist}_{\mathrm{H}}(\mathsf{S},\mathsf{K}).$$

First, let us show for each fixed $\boldsymbol{x}\in\mathsf{C}$ that

$$\mathrm{dist}(\boldsymbol{x};\mathsf{K}) \le \mathrm{dist}(\boldsymbol{x};\mathsf{S}) + \max_{\boldsymbol{z}\in\mathsf{S}} \mathrm{dist}(\boldsymbol{z};\mathsf{K}).$$

Suppose thatthe minimum distance from the fixed $\boldsymbol{x}$ to the set $\mathsf{S}$ is attained at point $\boldsymbol{z}_\star\in\mathsf{S}$. Further suppose that the minimum distance from $\boldsymbol{z}_\star$ to the set $\mathsf{K}$ is attained at point $\boldsymbol{y}_\star\in\mathsf{K}$:

$$\begin{aligned}\mathrm{dist}(\boldsymbol{x};\mathsf{S}) &= \min_{\boldsymbol{z}\in\mathsf{S}} \|\boldsymbol{x}-\boldsymbol{z}\| = \|\boldsymbol{x}-\boldsymbol{z}_\star\|\\ \mathrm{dist}(\boldsymbol{z}_\star;\mathsf{K}) &= \min_{\boldsymbol{y}\in\mathsf{K}} \|\boldsymbol{z}_\star-\boldsymbol{y}\| = \|\boldsymbol{z}_\star-\boldsymbol{y}_\star\|.\end{aligned}$$

Then we have

$$\begin{aligned}\mathrm{dist}(\boldsymbol{x};\mathsf{K}) &\le \|\boldsymbol{x}-\boldsymbol{y}_\star\|\\ &\le \|\boldsymbol{x}-\boldsymbol{z}_\star\| + \|\boldsymbol{z}_\star-\boldsymbol{y}_\star\|\\ &= \mathrm{dist}(\boldsymbol{x},\mathsf{S}) + \mathrm{dist}(\boldsymbol{z}_\star,\mathsf{K})\\ &\le \mathrm{dist}(\boldsymbol{x},\mathsf{S}) + \max_{\boldsymbol{z}\in\mathsf{S}} \mathrm{dist}(\boldsymbol{z},\mathsf{K}).\end{aligned}$$

This computation follows from basic properties of the norm and the distance.

Second, we have

$$\begin{aligned}\mathrm{dist}(\boldsymbol{x};\mathsf{K}) &\le \mathrm{dist}(\boldsymbol{x};\mathsf{S}) + \max_{\boldsymbol{z}\in\mathsf{S}} \mathrm{dist}(\boldsymbol{z};\mathsf{K})\\ &\le \max_{\boldsymbol{x}_\star\in\mathsf{C}} \mathrm{dist}(\boldsymbol{x}_\star;\mathsf{S}) + \max_{\boldsymbol{z}\in\mathsf{S}} \mathrm{dist}(\boldsymbol{z},\mathsf{K})\\ &\le \max\left\{\max_{\boldsymbol{x}_\star\in\mathsf{C}} \mathrm{dist}(\boldsymbol{x}_\star;\mathsf{S});\ \max_{\boldsymbol{z}\in\mathsf{S}} \mathrm{dist}(\boldsymbol{z};\mathsf{C})\right\}\\ &\qquad + \max\left\{\max_{\boldsymbol{z}\in\mathsf{S}} \mathrm{dist}(\boldsymbol{z};\mathsf{K});\ \max_{\boldsymbol{y}\in\mathsf{K}} \mathrm{dist}(\boldsymbol{y};\mathsf{S})\right\}\\ &= \mathrm{dist}_{\mathrm{H}}(\mathsf{C};\mathsf{S}) + \mathrm{dist}_{\mathrm{H}}(\mathsf{S},\mathsf{K}).\end{aligned}$$

The last identity is the definition of the Hausdorff distance. Because this relationship holds for all fixed $\boldsymbol{x}\in\mathsf{C}$, it must hold for the distinguished $\boldsymbol{x}$ that yields the maximum distance to $\mathsf{K}$:

$$\max_{\boldsymbol{x}\in\mathsf{C}} \mathrm{dist}(\boldsymbol{x};\mathsf{K}) \le \mathrm{dist}_{\mathrm{H}}(\mathsf{C},\mathsf{S}) + \mathrm{dist}_{\mathrm{H}}(\mathsf{S},\mathsf{K}).$$

Last, we can repeat the argument above for the symmetric case to get

$$\max_{\boldsymbol{y}\in\mathsf{K}} \operatorname{dist}(\boldsymbol{y};\mathsf{C}) \le \operatorname{dist}_{\mathrm{H}}(\mathsf{K},\mathsf{S}) + \operatorname{dist}_{\mathrm{H}}(\mathsf{S},\mathsf{C}) = \operatorname{dist}_{\mathrm{H}}(\mathsf{C},\mathsf{S}) + \operatorname{dist}_{\mathrm{H}}(\mathsf{S},\mathsf{K}).$$

In combination,

$$\max\left\{\max_{\boldsymbol{x}\in\mathsf{C}} \operatorname{dist}(\boldsymbol{x};\mathsf{K});\ \max_{\boldsymbol{y}\in\mathsf{K}} \operatorname{dist}(\boldsymbol{y};\mathsf{C})\right\} \le \operatorname{dist}_{\mathrm{H}}(\mathsf{C},\mathsf{S}) + \operatorname{dist}_{\mathrm{H}}(\mathsf{S},\mathsf{K}).$$

But the left side of the inequality is just the definition of the Hausdorff distance between $\mathsf{C}$ and $\mathsf{K}$. Hence,

$$\operatorname{dist}_{\mathrm{H}}(\mathsf{C},\mathsf{K}) \le \operatorname{dist}_{\mathrm{H}}(\mathsf{C},\mathsf{S}) + \operatorname{dist}_{\mathrm{H}}(\mathsf{S},\mathsf{K}).$$

This establishes the triangle inequality.

□

### 7.6.2 The Limit of Nonempty Sets is Nonempty

Last, we demonstrate that the Hausdorff limit of a sequence of nonempty convex bodies must be a nonempty convex body.

**Theorem 7.6.2** (Nonempty limit)**.** *Let $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3, \dots$ be a sequence of convex bodies that converges to $\mathsf{C}$ in the Hausdorff metric. If the $\mathsf{C}_i$ are nonempty for all $i \in \mathbb{N}$, then the limit $\mathsf{C}$ is also nonempty.*

*Proof.* The assumption implies that for all $\varepsilon > 0$, there exists an $I > 0$ such that, for all $i \ge I$, we have

$$\operatorname{dist}_{\mathrm{H}}(\mathsf{C}_i,\mathsf{C}) \le \varepsilon.$$

Suppose that the limit $\mathsf{C}$ were empty. This means that the Hausdorff distance between itself and any other convex body is infinite. This contradicts the fact that $\operatorname{dist}_{\mathrm{H}}(\mathsf{C}_i,\mathsf{C}) \le \varepsilon$ for all sets $\mathsf{C}_i$ with $i \ge I$. We conclude that $\mathsf{C}$ must be nonempty. □

# Lecture 8: Steiner's Formula

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*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*25 October 2018*

## 8.1 Agenda for Lecture 8

Let $\mathsf{B}_d$ denote the unit ball in $\mathbb{R}^d$. For a convex body $\mathsf{C}$ in $\mathbb{R}^d$ and some $\lambda \geq 0$, we define the *parallel body* $\mathsf{C}_\lambda := \mathsf{C} + \lambda \mathsf{B}_d$. This lecture is dedicated to understanding how the volume of a parallel body depends on $\lambda$. Specifically, we will show that the function $\lambda \mapsto S_{\mathsf{C}}(\lambda) := \mathrm{Vol}_d(\mathsf{C}_\lambda)$ is a degree-$d$ polynomial.

The coefficients of the polynomial $S_{\mathsf{C}}$ depend on the *intrinsic volumes* of the convex body $\mathsf{C}$. When $\mathsf{C}$ is a polytope, these intrinsic volumes have a rich geometric interpretation that connects to polyhedral tilings of $\mathbb{R}^d$, as addressed in the previous lecture. Understanding the intrinsic volumes in the polyhedral case is a key component in proving that $S_{\mathsf{C}}$ is a polynomial for arbitrary convex bodies.

1. Worked Examples in the Plane
2. Steiner's Formula for Polytopes
3. Understanding Intrinsic Volumes
4. Continuity of Intrinsic Volumes
5. Extending Steiner's Formula to Arbitrary Convex Bodies

## 8.2 Worked Examples in the Plane

Figure 8.1 shows two polytopes in the plane (a triangle $\mathsf{T}$ and a rectangle $\mathsf{R}$) along with their parallel bodies for a common value of $\lambda$.

In class, we constructed these parallel bodies by first translating a copy of each edge away from the set by distance $\lambda$, and then connecting the endpoints of "adjacent" translated edges by arcs.

For the triangle $\mathsf{T}$, the first of these steps contributed $+3\lambda L$ units of area, where $L$ was the length of a side of $\mathsf{T}$. The second of these steps contributed three identical "caps", each formed by cutting out a 120-degree slice of a circle with radius $\lambda$. Identifying $3L = \mathrm{perim}(\mathsf{T})$ as the perimeter of the triangle $\mathsf{T}$, we find that $S_{\mathsf{T}}(\lambda) = \mathrm{area}(\mathsf{T}) + \lambda\, \mathrm{perim}(\mathsf{T}) + \pi\lambda^2$.

For the rectangle $\mathsf{R}$, we similarly group all area contributions from the edges, and we see that they add $+\lambda\, \mathrm{perim}(\mathsf{R})$. The endcaps again contribute the area of a radius-$\lambda$ circle to the parallel body, which results in total area $S_{\mathsf{R}}(\lambda) = \mathrm{area}(\mathsf{R}) + \lambda\, \mathrm{perim}(\mathsf{R}) + \pi\lambda^2$.

As it happens, *every* polytope $\mathsf{P}$ in $\mathbb{R}^2$ has the property that the Steiner polynomial $S_{\mathsf{P}}(\lambda) = \mathrm{area}(\mathsf{P}) + \lambda\, \mathrm{perim}(\mathsf{P}) + \pi\lambda^2$.

## 8.3 Steiner's Formula for Polytopes

The elementary geometric reasoning used in the previous section does not readily generalize to higher dimensions. It seems as though we can do little more than restate the definition of

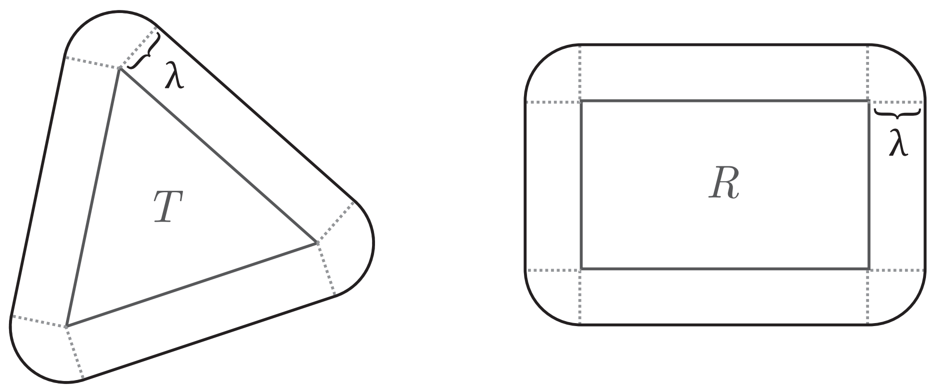


**Figure 8.1** (Some parallel bodies). The parallel bodies at distance $\lambda$ for a triangle $\mathsf{T}$ and a rectangle $\mathsf{R}$.

volume: if $\mathbb{1}_{\mathsf{A}}(\boldsymbol{x})$ is the 0–1 indicator that $\boldsymbol{x}$ belongs to a set $\mathsf{A}$, then certainly

$$S_{\mathsf{C}}(\lambda) = \mathrm{Vol}_d(\mathsf{C} + \lambda \mathsf{B}_d) = \int_{\boldsymbol{x}\in\mathbb{R}^d} \mathbb{1}_{\mathsf{C}+\lambda\mathsf{B}_d}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

But how does this help us? Over the course of this section, we will directly compute this integral for convex *polytopes*, which will lead to the following theorem.

**Theorem 8.3.1** (Steiner's Formula)**.** *Let* $\mathsf{P}$ *be a polytope in* $\mathbb{R}^d$*. Then, for all* $\lambda \geq 0$*,*

$$S_{\mathsf{P}}(\lambda) = \mathrm{Vol}_d(\mathsf{P} + \lambda \mathsf{B}_d) = \sum\nolimits_{j=0}^{d} \lambda^{d-j} \kappa_{d-j} V_j(\mathsf{P}).$$

*The geometric functionals* $V_j(\mathsf{P})$ *are called* intrinsic volumes*, and the constants* $\kappa_i := \mathrm{Vol}_i(\mathsf{B}_i)$ *do not depend on the polytope* $\mathsf{P}$*.*

Our proof of Steiner's Formula in this polyhedral case has two key components. First, we will appeal to Theorem 4.1 from Lecture 6 to partition the region of integration into the disjoint union

$$\mathbb{R}^d = \dot{\bigcup}_{\mathsf{F}\lhd\mathsf{P}} \big(\mathrm{relint}(\mathsf{F}) + \mathsf{N}(\mathsf{F})\big),$$

where $\mathsf{F} \lhd \mathsf{P}$ means "$\mathsf{F}$ is a face of $\mathsf{P}$" and $\mathsf{N}(\mathsf{F})$ is the normal cone of $\mathsf{F}$ with respect to $\mathsf{P}$. The polyhedral nature of $\mathsf{P}$ is important here because it ensures that the set $\{\mathsf{F} : \mathsf{F} \lhd \mathsf{P}\}$ of faces is finite. In addition, each face and its normal cone have complementary dimensions. The second step boils down to showing that, for any polytope $\mathsf{F}$ and for a large class of functions $f : \mathbb{R}_+ \to \mathbb{R}$, there is an especially simple expression for the integral

$$\int_{\mathrm{relint}(\mathsf{F})+\mathsf{N}(\mathsf{F})} f(\mathrm{dist}(\boldsymbol{x};\mathsf{F}))\,\mathrm{d}\boldsymbol{x}. \tag{8.3.1}$$

An expression for such an integral is useful to us because of the following fact: if $\boldsymbol{x}$ belongs to $\operatorname{relint}(\mathsf{F}) + \mathsf{N}(\mathsf{F})$ and $f = \mathbb{1}_{[0,\lambda]}$, then we can write $\mathbb{1}_{\mathsf{P}+\lambda\mathsf{B}_d}(\boldsymbol{x}) = f(\operatorname{dist}(\boldsymbol{x};\mathsf{F}))$. Our proof of Steiner's Formula then combines the these two ingredients by expressing

$$\int_{\mathbb{R}^d} \mathbb{1}_{\mathsf{P}+\lambda\mathsf{B}_d}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum\nolimits_{\mathsf{F}\lhd\mathsf{P}} \int_{\operatorname{relint}(\mathsf{F})+\mathsf{N}(\mathsf{F})} \mathbb{1}_{[0,\lambda]}(\operatorname{dist}(\boldsymbol{x};\mathsf{F}))\,\mathrm{d}\boldsymbol{x}. \tag{8.3.2}$$

Then we show that the right-hand side has the desired polynomial dependence on $\lambda$.

To improve the flow of our formal proof for Steiner's Formula, we first prove a lemma regarding integrals of the form (8.3.1). In what follows, $\mathbb{S}^{k-1}$ denotes the unit sphere in $\mathbb{R}^k$, and $\sigma_k$ is the measure[1] over $\mathbb{S}^{k-1}$ induced by the Lebesgue measure over $\mathbb{R}^k$. We call a function $f : \mathbb{R}_+ \to \mathbb{R}$ *light tailed* if the product $fq$ belongs to $L_1(\mathbb{R}_+)$ for every polynomial $q$.

**Lemma 8.3.2.** *For a $j$-dimensional face $\mathsf{F} \lhd \mathsf{P}$ of a polytope in $\mathbb{R}^d$ and a light-tailed function $f$ we have*

$$\begin{aligned} I &:= \int_{\operatorname{relint}(\mathsf{F})+\mathsf{N}(\mathsf{F})} f(\operatorname{dist}(\boldsymbol{x};\mathsf{F}))\,\mathrm{d}\boldsymbol{x} \\ &= \begin{cases} \operatorname{Vol}_j(\mathsf{F}) \cdot \sigma_{d-j}(\mathsf{N}(\mathsf{F})) \displaystyle\int_0^\infty f(r) r^{d-j-1}\,\mathrm{d}r, & j < d \\ \operatorname{Vol}_d(\mathsf{F}) \cdot f(0), & j = d. \end{cases} \end{aligned}$$

*Proof.* If $j = d$, then $\mathsf{N}(\mathsf{F}) = \{\mathbf{0}\}$, and so $\operatorname{dist}(\boldsymbol{x};\mathsf{F}) = 0$ for all $\boldsymbol{x}$ in the region of integration. Thus, when $j = d$, it is clear that $I = \operatorname{Vol}_d(\mathsf{F}) \cdot f(0)$.

Henceforth, consider $j < d$. Because $f$ is light tailed, it must be in $L_1(\mathbb{R}_+)$. Using the fact that $\operatorname{relint}(\mathsf{F})$ and $\mathsf{N}(\mathsf{F})$ are orthogonal, we can rewrite the integral $I$ as an iterated integral

$$I = \int_{\operatorname{relint}(\mathsf{F})} \mathrm{d}\boldsymbol{y} \int_{\mathsf{N}(\mathsf{F})} \mathrm{d}\boldsymbol{z}\, f(\operatorname{dist}(\boldsymbol{y}+\boldsymbol{z};\mathsf{F}))$$

where $\mathrm{d}\boldsymbol{y}$ and $\mathrm{d}\boldsymbol{z}$ are the Lebesgue measures on $\operatorname{aff}(\mathsf{F})$ and $\operatorname{aff}(\mathsf{N}(\mathsf{F}))$, respectively. If $\mathsf{F} = \{\boldsymbol{x}_0\}$ is a vertex (i.e., a 0-dimensional face), then the Lebesgue measure is the unit mass at the point $\boldsymbol{x}_0$.

Next, use the orthogonality of $\operatorname{relint}(\mathsf{F})$ and $\mathsf{N}(\mathsf{F})$ again to observe that $f(\operatorname{dist}(\boldsymbol{y}+\boldsymbol{z};\mathsf{F})) = f(\|\boldsymbol{z}\|)$ for all $(\boldsymbol{y},\boldsymbol{z}) \in \operatorname{relint}(\mathsf{F}) \times \mathsf{N}(\mathsf{F})$. This allows us to simplify the double integral to

$$I = \operatorname{Vol}_j(\mathsf{F}) \int_{\mathsf{N}(\mathsf{F})} \mathrm{d}\boldsymbol{z}\, f(\|\boldsymbol{z}\|).$$

The remaining integral is evaluated by changing to polar coordinates. In particular, the reader may verify that

$$\begin{aligned} \int_{\mathsf{N}(\mathsf{F})} \mathrm{d}\boldsymbol{z}\, f(\|\boldsymbol{z}\|) &= \int_{\mathsf{N}(\mathsf{F})\cap\mathbb{S}^{d-j-1}} \mathrm{d}\sigma_{d-j} \int_0^\infty \mathrm{d}r\, f(r) r^{d-j-1} \\ &= \sigma_{d-j}(\mathsf{N}(\mathsf{F})) \int_0^\infty f(r) r^{d-j-1} \mathrm{d}r, \end{aligned}$$

where the final integral is finite because $f$ is light-tailed. □

[1] If we apply $\sigma_k$ to some set $\mathsf{A} \subset \mathbb{R}^k$ that is *not* contained within the unit sphere, it should be understood that we mean $\sigma_k(\mathsf{A} \cap \mathbb{S}^{k-1})$.

With Lemma 8.3.2 in hand, we can give expressions for the intrinsic volumes that appear in Steiner's formula. Let $\omega_j = \sigma_j(\mathbb{S}^{j-1}) = j\kappa_j$. For a polytope $\mathsf{P}$, introduce the set of $j$-dimensional faces: $\mathscr{F}_j(\mathsf{P}) := \{\mathsf{F} \lhd \mathsf{P} : \dim \mathsf{F} = j\}$.

**Definition 8.3.3** (Intrinsic volumes of a polytope)**.** The $j$th *intrinsic volume* of a polytope $\mathsf{P} \subset \mathbb{R}^d$ is defined as

$$V_j(\mathsf{P}) = \sum\nolimits_{\mathsf{F} \in \mathscr{F}_j(\mathsf{P})} \mathrm{Vol}_j(\mathsf{F}) \measuredangle_{d-j}(\mathsf{N}(\mathsf{F})),$$

where $\measuredangle_{d-j}(\mathsf{N}(\mathsf{F})) := \sigma_{d-j}(\mathsf{N}(\mathsf{F}))/\omega_{d-j}$ is the proportion of the $(d-j-1)$-dimensional sphere subtended by the normal cone $\mathsf{N}(\mathsf{F})$.

We are now prepared to prove Steiner's formula for polytopes.

*Proof of Steiner's formula for polytopes.* Let $\mathsf{P}$ be a nonempty polytope in $\mathbb{R}^d$, and fix $\lambda \geq 0$. By Equation (8.3.2), we have

$$S_{\mathsf{P}}(\lambda) = \sum\nolimits_{j=0}^{d} \sum\nolimits_{\mathsf{F} \in \mathscr{F}_j(\mathsf{P})} \int_{\mathrm{relint}(\mathsf{F}) + \mathsf{N}(\mathsf{F})} \mathbb{1}_{[0,\lambda]}(\mathrm{dist}(\boldsymbol{x}; \mathsf{F}))\, \mathrm{d}\boldsymbol{x}.$$

The remaining integrals can be computed by Lemma 8.3.2 with $f = \mathbb{1}_{[0,\lambda]}$. For a $j$-dimensional face $\mathsf{F}$, this gives

$$\begin{aligned}\int_{\mathrm{relint}(\mathsf{F}) + \mathsf{N}(\mathsf{F})} \mathbb{1}_{[0,\lambda]}(\mathrm{dist}(\boldsymbol{x}; \mathsf{F}))\, \mathrm{d}\boldsymbol{x} &= \mathrm{Vol}_j(\mathsf{F}) \sigma_{d-j}(\mathsf{N}(\mathsf{F})) \frac{\lambda^{d-j}}{d-j} \\ &= \mathrm{Vol}_j(\mathsf{F}) \measuredangle_j(\mathsf{N}(\mathsf{F})) \lambda^{d-j} \frac{\omega^{d-j}}{d-j} \\ &= \mathrm{Vol}_j(\mathsf{F}) \measuredangle_j(\mathsf{N}(\mathsf{F})) \lambda^{d-j} \kappa_{d-j}.\end{aligned}$$

We then sum over the faces $\mathsf{F} \in \mathscr{F}_j(\mathsf{P})$ to reach

$$\sum\nolimits_{\mathsf{F} \in \mathscr{F}_j(\mathsf{P})} \int_{\mathrm{relint}(\mathsf{F}) + \mathsf{N}(\mathsf{F})} \mathbb{1}_{[0,\lambda]}(\mathrm{dist}(\boldsymbol{x}; \mathsf{F})) = \lambda^{d-j} \kappa_{d-j} V_j(\mathsf{P}).$$

Collect the terms of the sum over $j$ to arrive at Steiner's formula. □

## 8.4 Understanding Intrinsic Volumes

Having proved Steiner's Formula in the polyhedral case, let us discuss some properties of intrinsic volumes. In the following, $\mathsf{P}, \mathsf{P}' \in \mathscr{C}_d$ are nonempty polytopes, and the index $j$ ranges over $\{0, 1, 2, \dots, d\}$.

1. (Nonnegativity). Each intrinsic volume $V_j(\mathsf{P}) \geq 0$.

   Indeed, $\mathrm{Vol}_j$ is nonnegative, and $\measuredangle_j$ take values in $[0, 1]$.
2. (Volume). The intrinsic volume $V_d(\mathsf{P}) = \mathrm{Vol}_d(\mathsf{P})$.
3. (Euler characteristic). Every nonempty polytope $\mathsf{P}$ has $V_0(\mathsf{P}) = 1$.

   Equivalently, the normal cones of vertices of a polytope end up coving the entire unit sphere. This is consistent with the worked examples from Section 8.2. It follows in a roundabout way from Riedemeister's theorem. It can also be proved more directly using conjugate faces and polarity.

4. (Homogeneity). $V_j(\lambda \mathsf{P}) = \lambda^j V_j(\mathsf{P})$ for each $\lambda \geq 0$.

   The case $\lambda = 0$ is trivial. For $\lambda > 0$, this point follows because $\mathsf{F} \lhd \mathsf{P}$ if and only if $(\lambda \mathsf{F}) \lhd (\lambda \mathsf{P})$. The dilated face has $\mathrm{Vol}_j(\lambda \mathsf{F}) = \lambda^j \mathsf{F}$, while the normal cone of $\lambda \mathsf{F}$ in $\lambda \mathsf{P}$ coincides with the normal cone of $\mathsf{F}$ in $\mathsf{P}$.

5. (Monotonicity). If $\mathsf{P} \subset \mathsf{P}'$, then $V_j(\mathsf{P}) \leq V_j(\mathsf{P}')$.

   This is not easy to prove directly. It follows, for example, from Kubota's projection formula or Crofton's formula.

6. (Invariance). If $\boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^d$ is a rigid motion, then $V_j(\boldsymbol{T}\mathsf{P}) = V_j(\mathsf{P})$. Recall that a rigid motion is the composition of a translation and a rotation.

   This point is a straightforward consequence of the definitions of faces, normal cones, and intrinsic volumes.

7. (Intrinsic). Intrinsic volumes are unaffected when a polytope is lifted into a higher dimensional space: $V_j(\mathsf{P} \times \{\mathbf{0}_i\}) = V_j(\mathsf{P})$.

   Indeed, if $\mathsf{F} \lhd \mathsf{P}$ is a $j$-dimensional face of $\mathsf{P}$, then the associated face of $\mathsf{Q} := \mathsf{P} \times \{\mathbf{0}_i\}$ is the set $\mathsf{F} \times \{\mathbf{0}_i\}$, while the normal cone is $\mathsf{N}_{\mathsf{Q}}(\mathsf{F} \times \{\mathbf{0}_i\}) = \mathsf{N}_{\mathsf{P}}(\mathsf{F}) \times \mathbb{R}^i$. The spherical angle $\measuredangle \mathsf{N}_{\mathsf{Q}}(\mathsf{F} \times \{\mathbf{0}_i\}) = \measuredangle \mathsf{N}_{\mathsf{P}}(\mathsf{F}) \cdot \kappa_{d-j}/\kappa_{d+i-j}$.

## 8.5 Continuity of Intrinsic Volumes

Extending intrinsic volumes to the non-polyhedral setting is a critical step in proving Steiner's formula for arbitrary convex bodies. A significant limitation of the polyhedral definition is that for non-polyhedral $\mathsf{C}$, the index set $\mathscr{F}_0(\mathsf{C})$ appearing in the definition of $V_0$ could be uncountable. The following definition bypasses this obstacle.

**Definition 8.5.1** (Intrinsic volumes of a convex body). Let $\mathsf{C}$ be a nonempty convex body in $\mathbb{R}^d$, and choose a sequence of polytopes $\{\mathsf{P}_n : n \in \mathbb{N}\}$ with Hausdorff limit $\lim_{n\to\infty} \mathsf{P}_n = \mathsf{C}$. Then we define the $j$th intrinsic volume of the convex body $V_j(\mathsf{C}) := \lim_{n\to\infty} V_j(\mathsf{P}_n)$.

For the above definition to be valid, we must prove that the limit exists and that it is independent of the particular sequence of approximating polytopes. Our proof will make use of the following proposition.

**Proposition 8.5.2** (Selectors). *There exist functions $f_0, \dots, f_d$ of the form $f_j(r) = q_j(r)\,\mathrm{e}^{-r}$ where $q_j$ are polynomials of degree at most $d$ that have the following properties.*

- $f_d(0) = 1$ *and* $\int_0^\infty f_d(r) r^i \,\mathrm{d}r = 0$ *for* $i = 0, 1, 2, \dots, d-1$.
- $f_j(0) = 0$ *and* $\int_0^\infty f_j(r) r^i \,\mathrm{d}r = \delta_{ij}$ *for* $i, j = 0, 1, 2, \dots, d-1$.

*Here, $\delta_{ij}$ is the Kronecker delta.*

The proposition essentially states that there exist continuous "light-tailed" functions that can grab the coefficient of a desired monomial appearing in a polynomial.

*Proof sketch.* Consider the linear space of polynomials with real coefficients and degree at most $d$. The linear functionals

$$\varphi_d : q \mapsto q(0) \quad \text{and} \quad \varphi_i : q \mapsto \int_0^\infty q(r) r^i \mathrm{e}^{-r} \,\mathrm{d}r \quad \text{for } i = 0, 1, 2, \dots, d-1$$

are linearly independent. Solve the linear system $\varphi_i(q_j) = \delta_{ij}$ to obtain the required functions $q_j$. □

*Proof (The intrinsic volumes are well-defined).* Fix a sequence of polytopes $\{\mathsf{P}_n : n \in \mathbb{N}\}$ that converges to $\mathsf{C}$ in the Hausdorff metric. Our immediate goal is to obtain an expression for $V_j(\mathsf{P}_n)$ other than the one given in its definition. Toward this end, we invoke Lemma 8.3.2. If $\mathsf{F}$ is a face of $\mathsf{P}_n$ with dimension less than $d$, then applying the lemma with $\mathsf{F}$ and $f_{d-j-1}$ gives

$$\int_{\operatorname{relint}(\mathsf{F})+\mathsf{N}(\mathsf{F})} f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{F}))\,\mathrm{d}\boldsymbol{x} = \begin{cases} \operatorname{Vol}_j(\mathsf{F})\sigma_{d-j}(\mathsf{N}(\mathsf{F})), & \dim \mathsf{F} = j \\ 0, & \dim \mathsf{F} \neq j. \end{cases}$$

We then sum these expressions over all faces $\mathsf{F}$ with dimension $j$ to obtain

$$\int_{\mathbb{R}^d} f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{P}_n))\,\mathrm{d}\boldsymbol{x} = \sum\nolimits_{\mathsf{F}\in\mathscr{F}_j(\mathsf{P}_n)} \operatorname{Vol}_j(F)\sigma_{d-j}(\mathsf{N}(\mathsf{F})) = \omega_{d-j} V_j(\mathsf{P}_n).$$

This yields an alternative expression for $V_j(\mathsf{P}_n)$:

$$V_j(\mathsf{P}_n) = \frac{1}{\omega_{d-j}} \int_{\mathbb{R}^d} f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{P}_n))\,\mathrm{d}\boldsymbol{x}. \tag{8.5.1}$$

The final thrust of the proof requires the following technical observations about the integrand in (8.5.1):

1. Since $\{\mathsf{P}_n : n \in \mathbb{N}\}$ converges to $\mathsf{C}$, each set is "well inside" some fixed Euclidean ball $\mathsf{B}$. Furthermore, there exist constants $c_1$ and $c_2$ with

$$|f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{P}_n))| \leq c_1 \exp(-c_2\|\boldsymbol{x}\|).$$

2. Continuity of $f_{d-j-1}$ over $\mathbb{R}_+$ and $\operatorname{dist}(\boldsymbol{x};\cdot)$ over convex bodies in $\mathbb{R}^d$ ensures that we have the pointwise limit

$$\lim_{n\to\infty} f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{P}_n)) = f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{C})) \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^d.$$

   In particular, these limits hold for *any* sequence of polytopes that converges to $\mathsf{C}$.

Invoking these properties in the order given, we can compute the limit

$$\begin{aligned}
\lim_{n\to\infty} V_j(\mathsf{P}_n) &= \frac{1}{\omega_{d-j}} \lim_{n\to\infty} \int_{\mathbb{R}^d} f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{P}_n))\,\mathrm{d}\boldsymbol{x} \\
&= \frac{1}{\omega_{d-j}} \int_{\mathbb{R}^d} \lim_{n\to\infty} f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{P}_n))\,\mathrm{d}\boldsymbol{x} \\
&= \frac{1}{\omega_{d-j}} \int_{\mathbb{R}^d} f_{d-j-1}(\operatorname{dist}(\boldsymbol{x};\mathsf{C}))\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

This limit clearly does not depend on the sequence of polytopes $\{\mathsf{P}_n\}$, so the intrinsic volume $V_j(\mathsf{C})$ is well-defined.

For the case $j = d$, a similar argument shows that

$$\lim_{n\to\infty} V_d(\mathsf{P}_n) = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_d(\operatorname{dist}(\boldsymbol{x}; \mathsf{P}_n))\, \mathrm{d}\boldsymbol{x}$$
$$= \int_{\mathbb{R}^d} \lim_{n\to\infty} f_d(\operatorname{dist}(\boldsymbol{x}; \mathsf{P}_n))\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} f_d(\boldsymbol{x}; \mathsf{C})\, \mathrm{d}\boldsymbol{x}.$$

We conclude that the intrinsic volume $V_d(\mathsf{C})$ is also well-defined. □

This section's last theorem was stated in class. This result is not required to extend Steiner's formula to convex bodies, but it will be important for the subsequent discussion.

**Theorem 8.5.3** (Continuity of intrinsic volumes). *The intrinsic volumes $V_j$ of a convex body, given by Definition 8.5.1, are continuous with respect to the Hausdorff metric.*

*Proof.* Let $\mathsf{C}$ be a convex body in $\mathbb{R}^d$, and suppose that $\{\mathsf{C}_n : n \in \mathbb{N}\}$ is a sequence of convex bodies in $\mathbb{R}^d$ that converges to $\mathsf{C}$. We will show that, for any $j \le d$,

$$\lim_{n\to\infty} V_j(\mathsf{C}_n) = V_j(\mathsf{C}).$$

For each index $n$, the definition of the intrinsic volume $V_j(\mathsf{C}_n)$ ensures that there exists some polytope $\mathsf{P}_n$ satisfying

$$|V_j(\mathsf{P}_n) - V_j(\mathsf{C}_n)| \le 1/n \quad \text{and} \quad \operatorname{dist}_{\mathrm{H}}(\mathsf{P}_n, \mathsf{C}_n) \le 1/n.$$

These polytopes form a sequence $\{\mathsf{P}_n : n \in \mathbb{N}\}$. By uniqueness of Hausdorff limits and the fact that $\{\mathsf{C}_n : n \in \mathbb{N}\}$ converges to $\mathsf{C}$, it must hold that $\mathsf{P}_n \to \mathsf{C}$. Now, since $\{\mathsf{P}_n : n \in \mathbb{N}\}$ is a sequence of *polytopes* converging to $\mathsf{C}$, Definition 8.5.1 ensures that we can express $V_j(\mathsf{C}) = \lim_{n\to\infty} V_j(\mathsf{P}_n)$. Then using $|V_j(\mathsf{C}) - \lim_{n\to\infty} V_j(\mathsf{C}_n)| = \lim_{n\to\infty} |V_j(\mathsf{P}_n) - V_j(\mathsf{C}_n)| = 0$, we arrive at the desired result. □

Warning 8.5.4 (Interpretation of intrinsic volumes). The combinatorial aspects of $V_j$ from Definition 8.3.3 *do not extend* to general convex bodies. Thus, the continuity of intrinsic volumes should not be construed as implying statements about the facial structure of a non-polyhedral convex body. Nevertheless, the interpretations of $V_d$ as the ordinary volume and $V_0$ as the Euler characteristic persist.

## 8.6 Extending Steiner's Formula to Arbitrary Convex Bodies

Finally, we are prepared to extend Steiner's formula to all convex bodies.

**Theorem 8.6.1** (Steiner's Formula). *Let $\mathsf{C} \subset \mathbb{R}^d$ be a convex body in $\mathbb{R}^d$. Then, for all $\lambda \ge 0$,*

$$S_{\mathsf{C}}(\lambda) = \operatorname{Vol}_d(\mathsf{C} + \lambda \mathsf{B}_d) = \sum\nolimits_{j=0}^{d} \lambda^{d-j} \kappa_{d-j} V_j(\mathsf{C}).$$

*The intrinsic volumes $V_i(\mathsf{C})$ are specified by Definition 8.5.1, and the constants $\kappa_i := \operatorname{Vol}_i(\mathsf{B}_i)$ are independent of the set $\mathsf{C}$.*

*Proof.* Let $\{\mathsf{P} : n \in \mathbb{N}\}$ be a sequence of polytopes converging to the convex body $\mathsf{C} \subset \mathbb{R}^d$. For each $n$, consider the polynomial $S_n(\lambda) = \mathrm{Vol}_d(\mathsf{P}_n + \lambda \mathsf{B}_d)$. The function $S_\infty = \lim_{n\to\infty} S_n$ is a well-defined polynomial, taking values

$$\begin{aligned} S_\infty(\lambda) &= \lim_{n\to\infty} \sum\nolimits_{j=0}^{d} \lambda^{d-j} \kappa_{d-j} V_j(\mathsf{P}_n) \\ &= \sum\nolimits_{j=0}^{d} \lambda^{d-j} \kappa_{d-j} \lim_{n\to\infty} V_j(\mathsf{P}_n) = \sum\nolimits_{j=0}^{d} \lambda^{d-j} \kappa_{d-j} V_j(\mathsf{C}). \end{aligned}$$

From here, we need only cite continuity of $\mathrm{Vol}_d = V_d$ to see that $S_\infty(\lambda) = S_{\mathsf{C}}(\lambda)$ to conclude on the desired result. □

There are a great many theorems similar to the one above that go beyond the volume of parallel bodies. In some cases these theorems refer to different measures of content; that is, they concern scalar functions $\lambda \mapsto \mu(\mathsf{C} + \lambda \mathsf{B}_d)$ for $\mu$ other than $\mathrm{Vol}_d$. In the next lecture, we lay the groundwork for analyzing some of these alternative measures of content.

Remark 8.6.2 (Distance integrals). By integrating Steiner's formula, we can obtain a general expression for the integral of the distance to a convex set. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be an integrable function. Then

$$\int_{\mathbb{R}^d} f(\mathrm{dist}(\boldsymbol{x}; \mathsf{C}))\, \mathrm{d}\boldsymbol{x} = f(0) \cdot V_d(\mathsf{C}) + \sum\nolimits_{j=0}^{d-1} \left( \int_0^\infty f(r) r^{d-j-1}\, \mathrm{d}r \right) \omega_{d-j} \cdot V_j(\mathsf{C}).$$

This expression is valid whenever the integrals on the right-hand side are all finite. We omit the proof.

# Lecture 9: Valuations

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*ACM 204, Fall 2018*
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*30 October 2018*

## 9.1 Agenda for Lecture 9

There are many interesting functionals on sets that measure different types of "content." Examples include the ordinary volume (i.e., Lebesgue measure), the Gaussian measure, or the number of integer-valued vectors in a given set. All of these examples derive from measures. In convex geometry, it is natural to consider a more combinatorial approach to the notion of content, rather than the measure-theoretic approach that you are familiar with. To that end, we will study two different types of *valuations*, both of which abstract the basic additivity property of a measure.

In the first part of the lecture, we rigorously introduce the concept of a *set valuation*. Set valuations are defined on certain collections of sets, called *intersectional families*. Then we develop a few simple (but nontrivial) identities for the Minkowski sum. These identities allow us to prove that the intrinsic volumes are valuations.

In the second half of the lecture, we discuss how to extend set valuations to larger families of sets that also include unions. To do so, it is more convenient to work in a linear space of indicator functions generated by sets. This change of perspective turns out to be simpler than remaining in the world of sets. In this more general environment, we define the concept of a *linear valuation*, and we discuss how linear valuations induce set valuations. Finally, we establish Groemer's extension theorem, which describes circumstances when set valuations can be used to construct linear valuations.

1. Set valuations
2. Identities for Minkowski sum
3. Examples of set valuations
4. The algebra of sets
5. Linear valuations
6. Groemer's extension theorem

## 9.2 Set Valuations

We begin with the familiar notion of the volume of a set in $\mathbb{R}^d$. One of the fundamental properties of volume is the additivity law

$$\mathrm{Vol}_d(\emptyset) = 0 \quad \text{and} \quad \mathrm{Vol}_d(\mathsf{A} \cup \mathsf{B}) = \mathrm{Vol}_d(\mathsf{A}) + \mathrm{Vol}_d(\mathsf{B}) - \mathrm{Vol}_d(\mathsf{A} \cap \mathsf{B}).$$

This identity holds for all (Borel) measurable sets $\mathsf{A}, \mathsf{B} \subset \mathbb{R}^d$.

Our goal is to define a set valuation, which is a notion of content that generalizes the additivity property of volume. We can assign a volume to every measurable set. Similarly, a set valuation will assign a number to each set from a distinguished class.

**Definition 9.2.1** (Intersectional family)**.** A class $\mathscr{S}$ of sets is *intersectional* if $\emptyset \in \mathscr{S}$ and if $\mathsf{A}, \mathsf{B} \in \mathscr{S}$ implies $\mathsf{A} \cap \mathsf{B} \in \mathscr{S}$.

**Example 9.2.2** (Intersectional families)**.** There are many fundamental examples of intersectional families of sets:

1. The collection of parallelotopes (that is, axis aligned rectangles) in $\mathbb{R}^d$.
2. The collection of polytopes in $\mathbb{R}^d$.
3. The collection of compact, convex sets in $\mathbb{R}^d$.
4. The collection of all convex sets in $\mathbb{R}^d$.
5. The class of all Borel sets in $\mathbb{R}^d$.

We remark that these families of sets are listed in increasing order.

We are now prepared to define the concept of a set valuation.

**Definition 9.2.3** (Set valuation)**.** A (real-valued) *set valuation* on an intersectional family $\mathscr{S}$ is a map $\mu : \mathscr{S} \to \mathbb{R}$ which satisfies

1. $\mu(\emptyset) = 0$;
2. $\mu(\mathsf{A} \cup \mathsf{B}) + \mu(\mathsf{A} \cap \mathsf{B}) = \mu(\mathsf{A}) + \mu(\mathsf{B})$ when $\mathsf{A}, \mathsf{B}, \mathsf{A} \cup \mathsf{B} \in \mathscr{S}$.

Note the requirement that the union $\mathsf{A} \cup \mathsf{B}$ is a member of $\mathscr{S}$!

Every measure $\mu$ gives a set valuation on the $\sigma$-algebra where it is defined. This observation leads to a large class of examples, but there are other types of set valuations as well.

**Example 9.2.4** (Set valuations)**.** As mentioned before, the parallelotopes in $\mathbb{R}^d$, the polytopes in $\mathbb{R}^d$, and the convex bodies in $\mathbb{R}^d$ each compose an intersectional family. For each one of these families $\mathscr{S}$, the following functions are set valuations:

1. **(Volume $\mathrm{Vol}_d$).** The Lebesgue measure $\mathrm{Vol}_d$ is a set valuation.
2. **(Gaussian measure $\gamma_d$).** The probability $\gamma_d(\mathsf{C})$ that a standard normal vector in $\mathbb{R}^d$ lies in a given set $\mathsf{C} \in \mathscr{S}$ is a set valuation.
3. **(Number of integer-valued vectors.)** We can define a set valuation $\mathsf{C} \mapsto \#(\mathsf{C} \cap \mathbb{Z}^d)$ that counts how many integer-valued vectors lie in a set $\mathsf{C} \in \mathscr{S}$.
4. **(Euler characteristic $V_0$).** The *Euler characteristic* is the set valuation

$$V_0(\mathsf{C}) := \begin{cases} 1, & \mathsf{C} \text{ is a nonempty convex body;} \\ 0, & \mathsf{C} \text{ is empty.} \end{cases}$$

   Although this example may seem trivial, some incredible consequences will emerge in the next lecture.

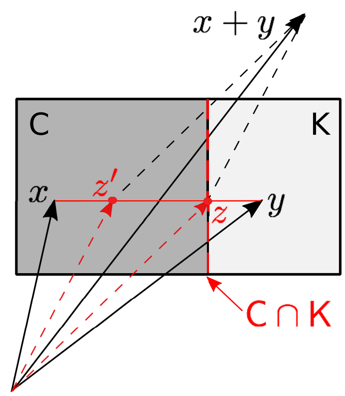


**Figure 9.1** (Minkowski sum identities). Suppose that $\mathsf{C}, \mathsf{K}, \mathsf{C} \cup \mathsf{K}$ are nonempty convex bodies. If $\boldsymbol{x} \in \mathsf{C}$ and $\boldsymbol{y} \in \mathsf{K}$, then the segment $[\boldsymbol{x}, \boldsymbol{y}]$ contains a point in the intersection $\mathsf{C} \cap \mathsf{K}$. In particular, the intersection is a nonempty convex body.

## 9.3 Identities for Minkowski Sum

Let us pause for a moment to establish some identities for the Minkowski sum. These results serve as a powerful tool for identifying set valuations on closed convex sets.

**Proposition 9.3.1.** *Assume that* $\mathsf{C}$, $\mathsf{K}$ *and* $\mathsf{C} \cup \mathsf{K}$ *are closed convex sets. For a closed convex set* $\mathsf{E}$*, the following identities hold.*

1. $(\mathsf{C} \cup \mathsf{K}) + (\mathsf{C} \cap \mathsf{K}) = \mathsf{C} + \mathsf{K}$.
2. $(\mathsf{C} \cup \mathsf{K}) + \mathsf{E} = (\mathsf{C} + \mathsf{E}) \cup (\mathsf{K} + \mathsf{E})$.
3. $(\mathsf{C} \cap \mathsf{K}) + \mathsf{E} = (\mathsf{C} + \mathsf{E}) \cap (\mathsf{K} + \mathsf{E})$.

Although these properties seem very basic, they were discovered quite late in the history of this subject. The result (1) first appeared in a 1966 paper of Sallee. We begin with a lemma that isolates the core of the argument.

**Lemma 9.3.2.** *Let* $\mathsf{C}, \mathsf{K}$ *be nonempty, closed convex sets in* $\mathbb{R}^d$*, and assume that* $\mathsf{C} \cup \mathsf{K}$ *is also convex. Choose a point* $\boldsymbol{x} \in \mathsf{C}$ *and a point* $\boldsymbol{y} \in \mathsf{K}$*. Then the segment* $[\boldsymbol{x}, \boldsymbol{y}]$ *has a nontrivial intersection with* $\mathsf{C} \cap \mathsf{K}$*. In particular,* $\mathsf{C} \cap \mathsf{K}$ *is a nonempty, closed convex set.*

*Proof.* Parameterize the segment $[\boldsymbol{x}, \boldsymbol{y}] = (1-\tau)\boldsymbol{x} + \tau\boldsymbol{y}$ for $\tau \in [0, 1]$. Let $\tau_{\mathsf{C}}$ be the maximum value of $\tau$ where $(1-\tau)\boldsymbol{x} + \tau\boldsymbol{y} \in \mathsf{C}$. Similarly, let $\tau_{\mathsf{K}}$ be the minimum value of $\tau$ where $(1-\tau)\boldsymbol{x} + \tau\boldsymbol{y} \in \mathsf{K}$. Both extrema are achieved because $\mathsf{C}, \mathsf{K}$ are closed and $[\boldsymbol{x}, \boldsymbol{y}]$ is bounded.

Suppose that $\tau_{\mathsf{C}} \neq \tau_{\mathsf{K}}$. Then we can interpose another number $\tau_{\mathsf{C}} < \tau_0 < \tau_{\mathsf{K}}$. By construction, $(1-\tau_0)\boldsymbol{x} + \tau_0\boldsymbol{y}$ is neither in $\mathsf{C}$ nor in $\mathsf{K}$. Since $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{C} \cup \mathsf{K}$, this contradicts the assumption that $\mathsf{C} \cup \mathsf{K}$ is convex.

We conclude that the point $\boldsymbol{z} = (1-\tau_{\mathsf{C}})\boldsymbol{x} + \tau_{\mathsf{C}}\boldsymbol{y} = (1-\tau_{\mathsf{K}})\boldsymbol{x} + \tau_{\mathsf{K}}\boldsymbol{y}$ belongs to both $\mathsf{C}$ and $\mathsf{K}$. □

*Proof of Proposition 9.3.1.* Property (2) is trivial, and property (3) follows the same idea as the proof of property (1). Therefore, we only present the proof of (1). The reader may wish to complete the argument as an exercise.

First, we check the easy inclusion: $(\mathsf{C}\cup\mathsf{K})+(\mathsf{C}\cap\mathsf{K}) \subset \mathsf{C}+\mathsf{K}$. Let $\boldsymbol{y} \in \mathsf{C}\cup\mathsf{K}$ and $\boldsymbol{z} \in \mathsf{C}\cap\mathsf{K}$. It is always possible to choose $\boldsymbol{z}$ in the opposite set from $\boldsymbol{y}$. Therefore, $\boldsymbol{y}+\boldsymbol{z} \in \mathsf{C}+\mathsf{K}$.

Next, we establish the more challenging inclusion: $\mathsf{C}+\mathsf{K} \subset (\mathsf{C}\cup\mathsf{K})+(\mathsf{C}\cap\mathsf{K})$. Let $\boldsymbol{x} \in \mathsf{C}$ and $\boldsymbol{y} \in \mathsf{K}$. Lemma 9.3.2 states that the segment $[\boldsymbol{x}, \boldsymbol{y}]$ contains a point $\boldsymbol{z} \in \mathsf{C}\cap\mathsf{K}$. Write $\boldsymbol{z} = (1-\tau)\boldsymbol{x} + \tau\boldsymbol{y}$. The point $\boldsymbol{z}' = \tau\boldsymbol{x} + (1-\tau)\boldsymbol{y} \in \mathsf{C}\cup\mathsf{K}$ because $\mathsf{C}\cup\mathsf{K}$ is convex. The relation $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{z}+\boldsymbol{z}'$ establishes the inclusion. Consult Figure 9.1 for an illustration. □

## 9.4 Examples of Set Valuations

With the help of the results in the previous section, we can identify some more valuations on the class of compact convex sets in $\mathbb{R}^d$.

### 9.4.1 Minkowski Additive Set Functions

First, we establish a general result which shows that any *additive* functional yields a set valuation.

**Definition 9.4.1** (Minkowski additive set function)**.** Let $\mathscr{S}$ be a family of sets in $\mathbb{R}^d$. A function $\mu : \mathscr{S} \to \mathbb{R}$ is *(Minkowski) additive* if

1. $\mu(\emptyset) = 0$;
2. $\mu(\mathsf{A}+\mathsf{B}) = \mu(\mathsf{A}) + \mu(\mathsf{B})$ for all nonempty sets $\mathsf{A}, \mathsf{B} \in \mathscr{S}$.

**Corollary 9.4.2** (Additive functions are valuations)**.** *Let $\mathscr{S}$ be the intersectional family of closed convex sets, and let $\mu : \mathscr{S} \to \mathbb{R}$ be an additive set function. Then $\mu$ is a set valuation on $\mathscr{S}$.*

*Proof.* Property (1) in Definition 9.2.3 of a set valuation is true per assumption.

To verify property (2), assume that $\mathsf{A}, \mathsf{B}, \mathsf{A}\cup\mathsf{B}$ are nonempty, closed, and convex. Lemma 9.3.2 implies that $\mathsf{A}\cap\mathsf{B}$ is a nonempty, closed convex set. Therefore, the additivity property of $\mu$ implies that

$$\mu(\mathsf{A}\cup\mathsf{B}) + \mu(\mathsf{A}\cap\mathsf{B}) = \mu((\mathsf{A}\cup\mathsf{B}) + (\mathsf{A}\cap\mathsf{B})) = \mu(\mathsf{A}+\mathsf{B}).$$

We have also used identity (1) from Proposition (9.3.1) in the final relation. □

The support function is, perhaps, the most important Minkowski additive functions. Corollary 9.4.2 implies that it is a set valuation.

**Corollary 9.4.3** (The support function is a valuation)**.** *Fix a direction $\boldsymbol{s} \in \mathbb{R}^d$. For a convex body $\mathsf{C}$ in $\mathbb{R}^d$, the support function in direction $\boldsymbol{s}$ is defined by the relations*

$$h(\boldsymbol{s}; \mathsf{C}) := \max\{\langle \boldsymbol{s}, \boldsymbol{x}\rangle : \boldsymbol{x} \in \mathsf{C}\} \quad \textit{and} \quad h(\boldsymbol{s}; \emptyset) = 0.$$

*Then $h(\boldsymbol{s}; \cdot)$ is a set valuation on the class of convex bodies in $\mathbb{R}^d$.*

*Proof.* Let us verify that the support function is Minkowski additive. For nonempty convex bodies $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$,

$$\begin{aligned}\max\{\langle \boldsymbol{s}, \boldsymbol{x}\rangle : \boldsymbol{x} \in \mathsf{C} + \mathsf{K}\} &= \max\{\langle \boldsymbol{s}, \boldsymbol{y} + \boldsymbol{z}\rangle : \boldsymbol{y} \in \mathsf{C}, \boldsymbol{z} \in \mathsf{K}\} \\ &= \max\{\langle \boldsymbol{s}, \boldsymbol{y}\rangle : \boldsymbol{y} \in \mathsf{C}\} + \max\{\langle \boldsymbol{s}, \boldsymbol{z}\rangle : \boldsymbol{z} \in \mathsf{K}\}.\end{aligned}$$

An application of Corollary 9.4.2 completes the argument. □

Remark 9.4.4 (Empty sets). With the usual conventions of convex analysis, we would define $h(\boldsymbol{s}; \emptyset) = -\infty$. We have chosen the definition here to be compatible with the theory of valuations.

### 9.4.2 New Valuations from Minkowski Addition

There is another important class of valuations obtained from the composition of a valuation with Minkowski addition.

**Corollary 9.4.5** (Valuations from Minkowski addition)**.** *Let $\mu$ be a set valuation on the class $\mathscr{C}_d$ of convex bodies in $\mathbb{R}^d$. Fix a convex body $\mathsf{E} \in \mathscr{C}_d$. Then $\mu(\cdot + \mathsf{E})$ is also a set valuation on $\mathscr{C}_d$.*

*Proof.* If $\mathsf{E}$ is empty, then the statement is vacuously true. Suppose instead that $\mathsf{E}$ is nonempty.

To verify property (1) of a set valuation, note that $\emptyset + \mathsf{E} = \emptyset$. Therefore, $\mu(\emptyset + \mathsf{E}) = \mu(\emptyset) = 0$.

To verify property (2), assume that $\mathsf{C}, \mathsf{K}, \mathsf{C} \cup \mathsf{K} \in \mathscr{C}_d$. Using the distributive identities (2) and (3) from Proposition 9.3.1, we obtain

$$\begin{aligned}\mu((\mathsf{C} \cup \mathsf{K}) + \mathsf{E}) + \mu((\mathsf{C} \cap \mathsf{K}) + \mathsf{E}) &= \mu((\mathsf{C} + \mathsf{E}) \cup (\mathsf{K} + \mathsf{E})) + \mu((\mathsf{C} + \mathsf{E}) \cap (\mathsf{K} + \mathsf{E})) \\ &= \mu(\mathsf{C} + \mathsf{E}) + \mu(\mathsf{K} + \mathsf{E}).\end{aligned}$$

The last relation follows because $\mu$ is a set valuation. □

An important consequence of Corollary 9.4.5 is that the intrinsic volumes are all set valuations.

**Corollary 9.4.6.** *Each intrinsic volume $V_j$ is a set valuation on the class $\mathscr{C}_d$ of convex bodies in $\mathbb{R}^d$.*

*Proof.* Fix an index $j = 0, 1, 2, \dots, d$. Recall that the volume $\mathrm{Vol}_d(\cdot)$ is a valuation on $\mathscr{C}_d$. For each $\lambda > 0$, Corollary 9.4.5 ensures that $\mathrm{Vol}_d(\cdot + \lambda \mathsf{B}_d)$ is a valuation. Therefore, when $\mathsf{C}, \mathsf{K}, \mathsf{C} \cup \mathsf{K} \in \mathscr{C}_d$,

$$\mathrm{Vol}_d((\mathsf{C} \cup \mathsf{K}) + \lambda \mathsf{B}_d) + \mathrm{Vol}_d((\mathsf{C} \cap \mathsf{K}) + \lambda \mathsf{B}_d) = \mathrm{Vol}_d(\mathsf{C} + \lambda \mathsf{B}_d) + \mathrm{Vol}_d(\mathsf{K} + \lambda \mathsf{B}_d). \quad (9.4.1)$$

Steiner's formula, from Lecture 8, states that

$$\mathrm{Vol}_d(\mathsf{E} + \lambda \mathsf{B}_d) = \sum\nolimits_{j=0}^{d} \lambda^{d-j} \kappa_{d-j} V_j(\mathsf{E}).$$

Applying this result to every term in equation (9.4.1) to see that

$$\sum_{j=0}^{d} \lambda^{d-j} \kappa_{d-j} \Big[ V_j(\mathsf{C} \cup \mathsf{K} + \lambda \mathsf{B}_d) + V_j(\mathsf{C} \cap \mathsf{K} + \lambda \mathsf{B}_d) - V_j(\mathsf{C} + \lambda \mathsf{B}_d) - V_j(\mathsf{K} + \lambda \mathsf{B}_d) \Big] = 0.$$

Since this polynomial vanishes for $\lambda > 0$, each of the coefficients is equal to zero. It follows that

$$V_j(\mathsf{C} \cup \mathsf{K} + \lambda \mathsf{B}_d) + V_j(\mathsf{C} \cap \mathsf{K} + \lambda \mathsf{B}_d) = V_j(\mathsf{C} + \lambda \mathsf{B}_d) + V_j(\mathsf{K} + \lambda \mathsf{B}_d).$$

Since $V_j(\emptyset) = 0$ by definition, we conclude that $V_j$ is a set valuation. □

## 9.5 The Algebra of Sets

We have been studying set valuations defined on intersectional families, such as the class of convex bodies in $\mathbb{R}^d$. We might also be interested in defining valuations for a larger family of sets. For instance, it is useful to be able to assign content to unions of convex bodies.

The most obvious approach to extending an intersectional family is to consider the smallest superset that is closed under unions. This method can be cumbersome because it involves inclusion–exclusion relations.

We will develop a different mechanism for extending set valuations from an intersectional family to a larger class. This approach replaces sets with their indicator functions.

**Definition 9.5.1** (Indicator function)**.** For a set $\mathsf{A} \subset \mathbb{R}^d$, the *0–1 indicator function* is the map $[\mathsf{A}] : \mathbb{R}^d \to \mathbb{R}$ defined by

$$[\mathsf{A}](\boldsymbol{x}) := \begin{cases} 1, & \boldsymbol{x} \in \mathsf{A} \\ 0, & \boldsymbol{x} \notin \mathsf{A}. \end{cases}$$

We also use the notation $\mathbb{1}_{\mathsf{A}}$ for the indicator, depending on typographical considerations.

Indicator functions interact very nicely with intersectional families because of the identity

$$[\mathsf{A} \cap \mathsf{B}] = [\mathsf{A}] \cdot [\mathsf{B}].$$

In other words, set intersection translates to pointwise multiplication of the indicator functions.

Given an intersectional family of sets, we can construct a linear space from the indicators.

**Definition 9.5.2** (Algebra of sets)**.** Let $\mathscr{S}$ be an intersectional family in $\mathbb{R}^d$. Introduce the real linear space generated by the indicators of sets from the intersectional family:

$$\mathbb{A}(\mathscr{S}) = \operatorname{lin}_{\mathbb{R}}\{[\mathsf{A}] : \mathsf{A} \in \mathscr{S}\}$$

This linear space is called the *algebra of sets* in $\mathscr{S}$ because $\mathbb{A}(\mathscr{S})$ is closed under real linear combinations and pointwise multiplication.

A generic element of the algebra $\mathbb{A}(\mathscr{S})$ takes the form

$$f = \sum_{i=1}^{m} \alpha_i [\mathsf{A}_i] \quad \text{where} \quad \alpha_i \in \mathbb{R} \quad \text{and} \quad \mathsf{A}_i \in \mathscr{S}.$$

That is, the algebra contains sums of indicator functions of sets in $\mathsf{S}$.

What does this buy us? For all sets $\mathsf{A}, \mathsf{B}$ in the intersectional family $\mathscr{S}$, we have the identity

$$[\mathsf{A}] + [\mathsf{B}] = [\mathsf{A} \cup \mathsf{B}] + [\mathsf{A} \cap \mathsf{B}]. \tag{9.5.1}$$

As a consequence, the indicator $[\mathsf{A} \cup \mathsf{B}]$ of the union belongs to the algebra $\mathbb{A}(\mathscr{S})$. Repeating this argument, we see that the algebra contains all finite unions of sets from the intersectional family.

## 9.6 Linear Valuations

As we have seen, the algebra contains the indicators of many new sets. Our next goal is to adapt the notion of a set valuation to this new setting. As we have seen, set valuations abstract the additive property of a measure. For an algebra, additive functions already play a central role in the guise of linear functionals.

**Definition 9.6.1** (Linear valuation)**.** A *linear valuation* on an intersectional family $\mathscr{S}$ is a linear functional on the algebra $\mathbb{A}(\mathscr{S})$.

Let us demonstrate that every linear valuation on the algebra $\mathbb{A}(\mathscr{S})$ induces a set valuation on the intersectional family $\mathscr{S}$. This result justifies the choice of terminology.

**Proposition 9.6.2.** *If $\varphi$ is a linear valuation $\mathscr{S}$, then the composition $\mathsf{A} \mapsto \varphi \circ [\mathsf{A}]$ defines a set valuation on $\mathscr{S}$.*

*Proof.* Since $\varphi$ is a linear functional and $[\emptyset] = -[\emptyset]$, we have $\varphi([\emptyset]) = -\varphi([\emptyset])$. Therefore, $\varphi([\emptyset]) = 0$. Next, assume that $\mathsf{A}, \mathsf{B} \in \mathscr{S}$. Since $\varphi$ is a linear functional,

$$\begin{aligned} \varphi([\mathsf{A}]) + \varphi([\mathsf{B}]) &= \varphi([\mathsf{A}] + [\mathsf{B}]) \\ &= \varphi([\mathsf{A} \cup \mathsf{B}] + [\mathsf{A} \cap \mathsf{B}]) = \varphi([\mathsf{A} \cup \mathsf{B}]) + \varphi([\mathsf{A} \cap \mathsf{B}]). \end{aligned}$$

We have used the identity (9.5.1) in the second step. □

## 9.7 Groemer's extension theorem

Conversely, we may ask if every set valuation $\mu$ on $\mathscr{S}$ induce a linear valuation $\varphi$ on $\mathbb{A}(\mathscr{S})$. It is natural to define the function $\varphi$ on the generating set of the algebra:

$$\varphi([\mathsf{A}]) := \mu(\mathsf{A}) \quad \text{for each } \mathsf{A} \in \mathscr{S}. \tag{9.7.1}$$

But it is not at all clear that we can extend this definition consistently to the entire algebra. Indeed, we require that

$$\varphi\left(\sum_{i=1}^{m} \alpha_i [\mathsf{A}_i]\right) = \sum_{i=1}^{m} \alpha_i \varphi([\mathsf{A}_i]) \quad \text{for all } \alpha_i \in \mathbb{R} \text{ and all } \mathsf{A}_i \in \mathscr{S}.$$

The problem is that elements of the algebra can have different descriptions. To ensure that $\varphi$ is a linear functional, we have to demonstrate that

$$\sum_{i=1}^{m} \alpha_i [\mathsf{A}_i] = \mathbf{0} \quad \text{implies that} \quad \sum_{i=1}^{m} \alpha_i \varphi([\mathsf{A}_i]) = 0. \tag{9.7.2}$$

We use the vector zero to denote the indicator of the empty set: $\mathbf{0} = [\emptyset]$. The condition (9.7.2) is probably not true for a general set valuation $\mu$ on an arbitrary intersectional family $\mathsf{S}$. The basic references on this subject, however, do not supply a counterexample.

Fortunately, in a number of important situations, it is possible to extend a set valuation to a linear valuation in the manner described above. The next theorem, due to Groemer, shows that every Hausdorff continuous set valuation on $\mathscr{C}_d$ extends to a linear valuation on $\mathbb{A}(\mathscr{C}_d)$.

**Theorem 9.7.1** (Groemer extension). *Every Hausdorff continuous valuation on the class $\mathscr{C}_d$ of convex bodies extends to a linear valuation on the algebra $\mathbb{A}(\mathscr{C}_d)$.*

*Proof.* Given a continuous set valuation $\mu$, we define a function $\varphi$ on the indicators of convex bodies via the rule (9.7.1): $\varphi([\mathsf{C}]) = \mu(\mathsf{C})$ for each $\mathsf{C} \in \mathscr{C}_d$. To ensure that $\varphi$ is well-defined, we must verify the condition (9.7.2).

The proof proceeds by induction on the dimension $d$. In case $d = 0$, there is nothing to prove because the algebra is one-dimensional. We will assume that the result holds for all continuous set valuations on $\mathscr{C}_{d-1}$, and we will establish the claim for $\mathscr{C}_d$.

The proof of the induction step is performed via contradiction. Assume that (9.7.2) fails. Then there exist $\alpha_i \in \mathbb{R}$ and $\mathsf{C}_i \in \mathscr{C}_d$ such that

$$\sum\nolimits_{i=1}^{m} \alpha_i [\mathsf{C}_i] = \mathbf{0}; \tag{9.7.3}$$

$$\sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i) = 1. \tag{9.7.4}$$

Among all such examples, select one where the number $m$ of terms is minimal.

Let $\mathsf{H}$ be a hyperplane whose closed halfspaces are denoted $\mathsf{H}^{\pm}$. Assume that $\mathsf{C}_1 \subset \operatorname{int}(\mathsf{H}^-)$. Since $[\mathsf{A} \cap \mathsf{B}] = [\mathsf{A}] \cdot [\mathsf{B}]$, the assumption (9.7.3) implies the following relations:

$$\sum\nolimits_{i=1}^{m} \alpha_i [\mathsf{C}_i \cap \mathsf{H}^-] = \mathbf{0}; \tag{9.7.5}$$

$$\sum\nolimits_{i=1}^{m} \alpha_i [\mathsf{C}_i \cap \mathsf{H}^+] = \mathbf{0}; \tag{9.7.6}$$

$$\sum\nolimits_{i=1}^{m} \alpha_i [\mathsf{C}_i \cap \mathsf{H}] = \mathbf{0}. \tag{9.7.7}$$

We now use the properties of the set valuation $\mu$ to derive related conclusions from the assumption (9.7.4). To that end, notice the following easy identities:

$$\mathsf{C}_i = (\mathsf{C}_i \cap \mathsf{H}^+) \cup (\mathsf{C}_i \cap \mathsf{H}^-) \quad \text{and} \quad \mathsf{C}_i \cap \mathsf{H} = (\mathsf{C}_i \cap \mathsf{H}^+) \cap (\mathsf{C}_i \cap \mathsf{H}^-).$$

Moreover, all the sets in these equations belong to the intersectional family $\mathscr{C}_d$, so $\mu$ is defined for each one. Since $\mu$ is a set valuation,

$$\mu((\mathsf{C}_i \cap \mathsf{H}^+) \cup (\mathsf{C}_i \cap \mathsf{H}^-)) + \mu((\mathsf{C}_i \cap \mathsf{H}^+) \cap (\mathsf{C}_i \cap \mathsf{H}^-)) = \mu(\mathsf{C}_i \cap \mathsf{H}^+) + \mu(\mathsf{C}_i \cap \mathsf{H}^-).$$

Equivalently,

$$\mu(\mathsf{C}_i) = \mu(\mathsf{C}_i \cap \mathsf{H}^+) + \mu(\mathsf{C}_i \cap \mathsf{H}^-) - \mu(\mathsf{C}_i \cap \mathsf{H}). \tag{9.7.8}$$

Plugging (9.7.8) into (9.7.4) yields

$$\begin{aligned}1 &= \sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i) \\ &= \sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{H}^+) + \sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{H}^-) - \sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{H}). \quad (9.7.9)\end{aligned}$$

The induction hypothesis ensures that $\mu$ induces a linear valuation on convex bodies in the hyperplane $\mathsf{H}$, so (9.7.7) implies that

$$\sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{H}) = 0.$$

By construction, $(\mathsf{C}_1 \cap \mathsf{H}^+) = \emptyset$, so the minimality of $m$ requires that

$$\sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{H}^+) = 0.$$

As a consequence of these two observations, (9.7.9) reduces to

$$\sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{H}^-) = 1.$$

This calculation can be repeated for other halfspaces and replacing $\mathsf{C}_1$ with other $\mathsf{C}_i$ to reach stronger conclusions.

We can always find a sequence $\{\mathsf{H}_j^-\}$ of halfspaces such that $\mathsf{C}_1 \subset \operatorname{int}(\mathsf{H}_j^-)$ for which the polyhedra $\mathsf{P}_q = \cap_{j=1}^{q} \mathsf{H}_j^-$ converge to $\mathsf{C}_1$ in the Hausdorff metric; cf. Lecture 7. For each halfspace $\mathsf{H}_j^-$, the foregoing considerations apply. Hence, for each index $q$, we obtain

$$\sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{H}_1^- \cdots \cap \mathsf{H}_q^-) = \sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{P}_q) = 1.$$

Since $\mu$ is continuous, we can take the limit as $q \to \infty$ to reach

$$\sum\nolimits_{i=1}^{m} \alpha_i \mu(\mathsf{C}_i \cap \mathsf{C}_1) = 1.$$

It follows directly from (9.7.5) that

$$\sum\nolimits_{i=1}^{m} \alpha_i [\mathsf{C}_i \cap \mathsf{C}_1] = \mathbf{0}.$$

We have shown that the assumptions (9.7.3) and (9.7.4) remain valid after intersection with the first set $\mathsf{C}_1$.

Repeat the same procedure with the remaining sets $\mathsf{C}_2, \mathsf{C}_3, \dots, \mathsf{C}_m$. This process yields the relations

$$\mathbf{0} = \sum\nolimits_{i=1}^{m} \alpha_i \Big[ \bigcap\nolimits_{j=1}^{m} \mathsf{C}_j \Big] = \Big( \sum\nolimits_{i=1}^{m} \alpha_i \Big) \Big[ \bigcap\nolimits_{j=1}^{m} \mathsf{C}_j \Big]; \quad (9.7.10)$$

$$1 = \sum\nolimits_{i=1}^{m} \alpha_i \mu \Big( \bigcap\nolimits_{j=1}^{m} \mathsf{C}_j \Big) = \Big( \sum\nolimits_{i=1}^{m} \alpha_i \Big) \mu \Big( \bigcap\nolimits_{j=1}^{m} \mathsf{C}_j \Big). \quad (9.7.11)$$

Now, the identity (9.7.11) implies that

$$\sum\nolimits_{i=1}^{m} \alpha_i \neq 0 \quad \text{and} \quad \mu \Big( \bigcap\nolimits_{j=1}^{m} \mathsf{C}_j \Big) \neq 0.$$

Since $\sum_{i=1}^m \alpha_i \neq 0$, the identity (9.7.10) forces us to conclude that

$$\bigcap_{j=1}^m \mathsf{C}_j = \emptyset.$$

A further application of (9.7.11) demonstrates that $\mu(\emptyset) \neq 0$ in contradiction to the fact that $\mu$ is a set valuation. $\square$

# Lecture 10: The Euler Characteristic

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*ACM 204, Fall 2018*
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*1 November 2018*

## 10.1 Agenda for Lecture 10

In this lecture, we will consider a very special valuation known as the Euler characteristic. It will allow us to measure or *evaluate* not just convex bodies, but also finite unions of convex bodies, which are generally non-convex. Applying this measurement tool to polytopes leads to an intriguing constraint on the distribution of their faces of different dimension. For a polytope in $\mathbb{R}^3$, this gives the famous formula that $\text{vertices} - \text{edges} + \text{facets} = 2$. Our agenda:

1. Recalls on valuations
2. The Euler characteristic
3. Hadwiger's construction
4. The Euler–Poincaré–Schläfli formula

## 10.2 Recalls on Valuations

We first recall the essential notion of an *intersectional* family. This term refers to a family of sets that is closed under finite intersections. More formally, it is defined as follows:

**Definition 10.2.1** (Intersectional family)**.** A class of sets $\mathscr{S}$ is *intersectional* if $\varphi \in \mathscr{S}$ and $\mathsf{A}, \mathsf{B} \in \mathscr{S}$ means that $\mathsf{A} \cap \mathsf{B} \in \mathscr{S}$.

Armed with these concepts, we remind the reader of the notion of a *set valuation*, which serves as a more abstract measure of volume.

**Definition 10.2.2** (Set valuation)**.** A real-valued set valuation on an intersectional family $\mathscr{S}$ is a map $\mu : \mathscr{S} \to \mathbb{R}$ satisfying

(1) $\mu(\varnothing) = 0$;

(2) $\mu(\mathsf{A} \cap \mathsf{B}) + \mu(\mathsf{A} \cup \mathsf{B}) = \mu(\mathsf{A}) + \mu(\mathsf{B})$ when $\mathsf{A}, \mathsf{B}, \mathsf{A} \cup \mathsf{B} \in \mathscr{S}$.

There are many examples of set valuations, such as the intrinsic volume.

**Example 10.2.3** (Intrinsic volumes are set valuations)**.** Each intrinsic volume $V_j$ is a set valuation on the class of convex bodies $\mathscr{C}_d$ living in $\mathbb{R}^d$.

At this point, we are confronted with a natural question: Can we extend set valuations to larger classes of sets? Suppose we can measure the "content" of a convex set $\mathsf{A}$, and we can also measure the "content" of a convex set $\mathsf{B}$. Can we measure the "content" of the union $\mathsf{A} \cup \mathsf{B}$, even though it is generally not convex?

To address this question, it is convenient to introduce functions in the *algebra of sets* in $\mathscr{S}$.

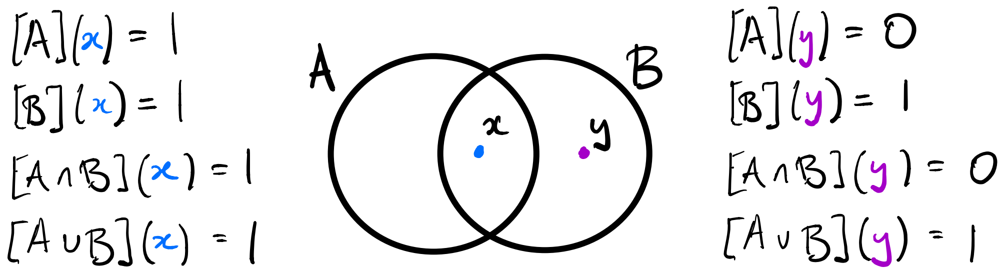


**Figure 10.1** (A simple Venn diagram). This figure is provided to ground the discussion. Inspection of the figure should convince the reader that indicators of set membership obey the law: $[\mathsf{A} \cup \mathsf{B}] + [\mathsf{A} \cap \mathsf{B}] = [\mathsf{A}] + [\mathsf{B}]$.

**Definition 10.2.4** (Algebra of sets)**.** Let $\mathscr{S}$ be an intersectional family in $\mathbb{R}^d$. The linear space

$$\mathbb{A}(\mathscr{S}) = \mathrm{lin}_{\mathbb{R}} \left\{ [\mathsf{A}] : \mathsf{A} \in \mathscr{S} \right\}$$

is called the *algebra of sets* in $\mathscr{S}$. Here, $[\mathsf{A}]$ denotes the 0–1 indicator of the set $\mathsf{A}$. A typical element of the algebra $\mathbb{A}(\mathscr{S})$ is a function of the form

$$f = \sum\nolimits_{i=1}^{m} \alpha_i [\mathsf{A}_i] \quad \text{for } \mathsf{A}_i \in \mathscr{S} \text{ and } \alpha_i \in \mathbb{R}.$$

As mentioned earlier, the algebra $\mathbb{A}(\mathscr{S})$ is convenient in our quest of extending valuations to non-convex unions. This is precisely because the algebra automatically contains all indicators of finite unions of sets in $\mathscr{S}$. To see this, note that $[\mathsf{A} \cup \mathsf{B}] = [\mathsf{A}] + [\mathsf{B}] - [\mathsf{A} \cap \mathsf{B}]$, which may be proved by staring at Figure 10.1. The implication of this statement is that the indicator $[\mathsf{A} \cup \mathsf{B}]$ is in the algebra $\mathbb{A}(\mathscr{S})$ because $\mathscr{S}$ is intersectional.

Now it is appropriate to recall the definition of a *linear valuation*.

**Definition 10.2.5** (Linear valuation)**.** A *linear valuation* is a linear functional $\varphi$ on the algebra of sets $\mathbb{A}(\mathscr{S})$.

For example, if $\varphi$ is a linear valuation, then

$$f = \sum\nolimits_i \alpha_i [\mathsf{A}_i] \quad \text{implies} \quad \varphi(f) = \sum\nolimits_i \alpha_i \varphi([\mathsf{A}_i]).$$

If $\mathsf{A}$ is a set whose indicator $[\mathsf{A}]$ belongs to the algebra $\mathbb{A}(\mathscr{S})$, we will sometimes drop the extra brackets when writing the linear functional: $\varphi(\mathsf{A}) := \varphi([\mathsf{A}])$.

The attentive reader may wonder: what justifies the name linear *valuation*? The answer is that all linear valuations give us set valuations. First note that $\varphi([\varnothing]) = \varphi(0) = 0$ by linearity. To check the second condition, we argue by linearity and the basic property of indicators:

$$\begin{aligned} \varphi([\mathsf{A}]) + \varphi([\mathsf{B}]) &= \varphi([\mathsf{A}] + [\mathsf{B}]) && \text{by linearity} \\ &= \varphi([\mathsf{A} \cup \mathsf{B}] + [\mathsf{A} \cap \mathsf{B}]) && \text{by Figure 10.1} \\ &= \varphi([\mathsf{A} \cup \mathsf{B}]) + \varphi([\mathsf{A} \cap \mathsf{B}]) && \text{by linearity.} \end{aligned}$$

Equating the first and last expression tells us that, indeed, the linear valuation $\varphi$ composed with the map $[\cdot]$ from sets to indicators is a set valuation.

To reiterate, we have shown that a linear valuation always extends to a set valuation:

$$\text{linear valuation} \quad \overset{\text{always}}{\Longrightarrow} \quad \text{set valuation}$$

But does the reverse hold?

$$\text{set valuation} \quad \overset{???}{\Longrightarrow} \quad \text{linear valuation}$$

Given a set valuation $\mu$ on $\mathscr{S}$, it would be natural to try to construct a linear functional by defining $\varphi([\mathsf{A}]) := \mu(\mathsf{A})$ for $\mathsf{A} \in \mathscr{S}$. The trouble is that it is not clear that this consistently extends to more complicated elements of the algebra. Indeed, we need to check that the extension is not one-to-many.

All is not lost, thankfully, and Volland and Groemer provided two results about when the extension does go through nicely. These theorems allow us to construct valuations on objects like unions of polytopes or unions of convex bodies.

**Theorem 10.2.6** (Volland). *Each set valuation on the class $\mathscr{P}_d$ of polytopes in $\mathbb{R}^d$ extends to a unique linear valuation on $\mathbb{A}(\mathscr{P}_d)$.*

**Theorem 10.2.7** (Groemer). *Each continuous set valuation on the class $\mathscr{C}_d$ of convex bodies in $\mathbb{R}^d$ extends to a unique linear valuation on $\mathbb{A}(\mathscr{C}_d)$.*

A consequence of Theorem 10.2.7 of particular interest is that we can define intrinsic volumes of unions of convex bodies. For example, we can give meaning to $V_j(\mathsf{A})$ when $\mathsf{A} = \mathsf{C}_1 \cup \ldots \cup \mathsf{C}_m$ for convex bodies $\mathsf{C}_i$.

Warning 10.2.8 (Extended intrinsic volumes). The nice geometric interpretations of the intrinsic volumes in the convex case are not always valid for the extension to the algebra of convex sets.

## 10.3 The Euler Characteristic

In this lecture, we will study the most innocuous example of a set valuation extended to the algebra of convex bodies: the Euler characteristic. Recall the definition of the zeroth intrinsic volume, $V_0$:

**Definition 10.3.1** (Zeroth intrinsic volume). Let $\mathsf{C} \in \mathscr{C}$ be a convex body. Then the zeroth intrinsic volume $V_0(\mathsf{C})$ is defined as

$$V_0(\mathsf{C}) = \begin{cases} 1, & \text{if } \mathsf{C} \text{ is nonempty;} \\ 0, & \text{if } \mathsf{C} \text{ is empty.} \end{cases}$$

To see that $V_0$ is a continuous valuation, we must only check that limits of nonempty convex bodies are nonempty. This follows from the fact that the Hausdorff distance between any nonempty set and the empty set is (defined to be) infinite, so nonempty sets cannot converge to empty ones.

With the continuity of $V_0$ established, we can apply Groemer's theorem to extend $V_0$ to a linear valuation on the algebra $\mathbb{A}(\mathscr{C}_d)$. This leads to the following (abstract) definition of the Euler characteristic.

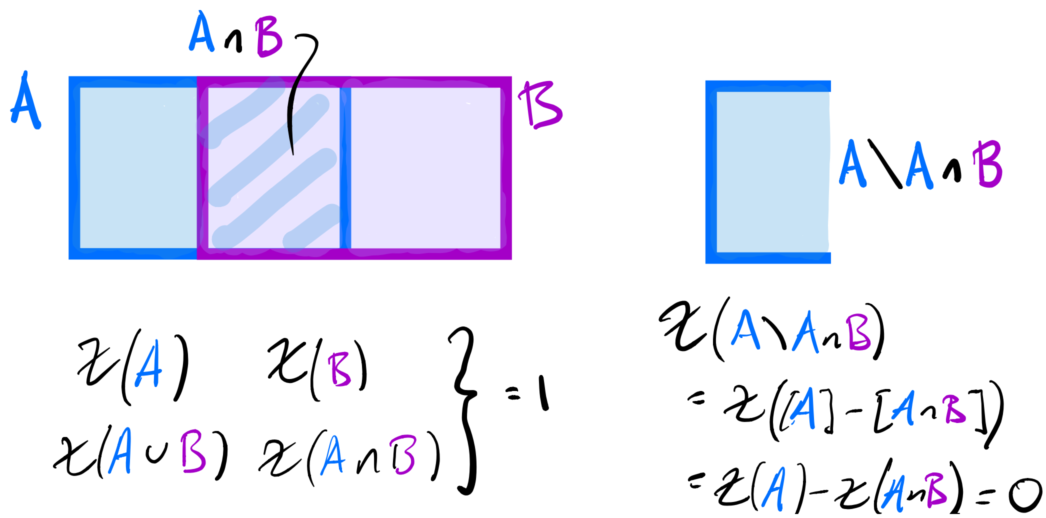

**Figure 10.2** (Example calculations of the Euler characteristic). [left] The Euler characteristic of non-empty convex bodies is trivially one. [right] More complicated arrangements may be derived by breaking down a set into constituent indicators. Removing a convex body decreases the Euler characteristic by one.

**Definition 10.3.2** (Euler characteristic)**.** The Euler characteristic $\chi : \mathbb{A}(\mathscr{C}_d) \to \mathbb{R}$ is the extension of the zeroth intrinsic volume $V_0$ to the algebra of convex bodies $\mathbb{A}(\mathscr{C}_d)$. In particular,

$$\chi\Big(\sum\nolimits_{i=1}^{m} \alpha_i[\mathsf{C}_i]\Big) = \sum\nolimits_{i=1}^{m} \alpha_i\chi([\mathsf{C}_i]) = \sum\nolimits_{\mathsf{C}_i\neq\varnothing} \alpha_i.$$

This definition is somewhat mysterious and abstract, so we give some example calculations in Figure 10.2. Here are a few exercises.

**Exercise 10.3.3.** Check that for the union of disjoint convex bodies, the Euler characteristic counts the number of bodies.

**Exercise 10.3.4.** Does the boundary of a polytope belong to the algebra of convex bodies? What about the relative interior?

**Exercise 10.3.5.** Does the open disk live in the algebra of convex bodies? If so, what is its Euler characteristic?

## 10.4 Hadwiger's Construction

In the previous section, we gave a somewhat abstract definition of the Euler characteristic motivated by Groemer's theorem, along with some computational examples. In this section, we will discuss a recursive construction of the Euler characteristic due to Hadwiger. The first part of the theorem will rehash the results of the previous section, but the second part of the theorem will go much further.

**Theorem 10.4.1** (Hadwiger)**.** *There is a unique linear valuation $\chi : \mathbb{A}(\mathscr{C}_d) \to \mathbb{R}$ on the algebra of convex bodies with the property that $\chi(\mathsf{C}) = 1$ if $\mathsf{C}$ is a nonempty convex body.*

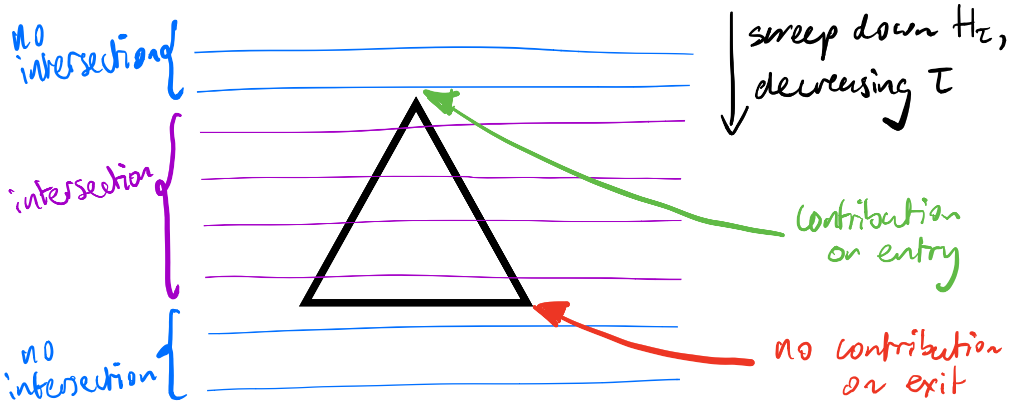


**Figure 10.3** (A visualisation of Hadwiger's construction). Hadwiger constructs the Euler characteristic by scanning a hyperplane along one coordinate axis. Summing "changes" in the Euler characteristics of the lower dimensional slices gives the Euler characteristic of the whole object. Note that we only register a contribution when the hyperplane enters a convex body, but not when it leaves.

*Moreover, if* $\mathsf{A} \subset \mathbb{R}^d$ *is a set for which* $[\mathsf{A}] \in \mathbb{A}(\mathscr{C}_d)$*, then the Euler characteristic of* $\mathsf{A}$ *may be computed by summing over discontinuous jumps in the Euler characteristic of slices of* $\mathsf{A}$ *swept across a coordinate axis:*

$$\chi(\mathsf{A}) = \sum\nolimits_{\tau \in \mathbb{R}} \Big[\chi(\mathsf{A} \cap \mathsf{H}_\tau) - \lim_{\varepsilon \downarrow 0} \chi(\mathsf{A} \cap \mathsf{H}_{\tau+\varepsilon})\Big], \tag{10.4.1}$$

*where* $\mathsf{H}_\tau := \{\boldsymbol{x} \in \mathbb{R}^d : x_d = \tau\}$ *is a hyperplane orthogonal to the dth coordinate axis.*

We provide a visualisation of Hadwiger's construction of the Euler characteristic in Figure 10.3. It will be helpful to keep this picture in mind during the following proof.

*Proof.* Uniqueness of the valuation $\chi$ is immediate. Indeed, the algebra $\mathbb{A}(\mathscr{C}_d)$ is spanned by the indicators of convex bodies, and we have defined the valuation for each convex body. The challenge is to show that the Euler characteristic is well-defined.

The proof will proceed by induction on dimension $d$. We begin with dimension $d = 0$. Each function $f$ in the algebra $\mathbb{A}(\mathscr{C}_0)$ must take the form $f = \alpha[\{\mathbf{0}\}]$ because the origin is the only point in $\mathbb{R}^0$. The definition $\chi(f) := \alpha$ produces a linear functional.

We now assume that there is a unique linear valuation $\chi : \mathbb{A}(\mathscr{C}_{d-1}) \to \mathbb{R}$ with the property that $\chi(\mathsf{C}) = 1$ for each nonempty convex body $\mathsf{C} \in \mathscr{C}_{d-1}$.

For any function $f \in \mathbb{A}(\mathscr{C}_d)$, define the restriction $f_\tau := f \cdot [\mathsf{H}_\tau]$ to the hyperplane at level $\tau$. It is clear that restriction is a linear map onto the subalgebra of functions that are supported on the hyperplane $\mathsf{H}_\tau$. This subalgebra is isomorphic to $\mathbb{A}(\mathscr{C}_{d-1})$, which will allow us to apply the induction hypothesis to restricted functions.

Express the function $f \in \mathbb{A}(\mathscr{C}_d)$ in the form

$$f = \sum\nolimits_{i=1}^{m} \alpha_i [\mathsf{C}_i] \quad \text{for } \mathsf{C}_i \in \mathscr{C}_d \text{ and } \alpha_i \in \mathbb{R}.$$

Observe that

$$f_\tau = \sum_{i=1}^{m} \alpha_i [\mathsf{C}_i \cap \mathsf{H}_\tau].$$

Since all of the functions are supported on $\mathsf{H}_\tau$, we can invoke the induction hypothesis to apply the Euler characteristic, which is a linear valuation on the lower-dimensional space:

$$\chi(f_\tau) = \sum_{i=1}^{m} \alpha_i \chi(\mathsf{C}_i \cap \mathsf{H}_\tau) = \sum_{\mathsf{C}_i \cap \mathsf{H}_\tau \neq \varnothing} \alpha_i.$$

From the above expression, it is clear that $\chi(f_\tau)$ only changes when the scanning hyperplane $\mathsf{H}_\tau$ leaves or enters one of the convex bodies $\mathsf{C}_i$.

We may now study the discontinuities in $\chi(f_\tau)$ as a function of $\tau$, the level of the hyperplane. Introduce the linear function $l : \boldsymbol{x} \mapsto x_d$ that returns the last coordinate of its input. Let $I$ index the convex bodies $\mathsf{C}_i$ which are just supported by and lie beneath $\mathsf{H}_\tau$. Then

$$\chi(f_\tau) - \lim_{\varepsilon \downarrow 0} \chi(f_{\tau+\varepsilon}) = \sum_{i \in I} \alpha_i \quad \text{where} \quad I = \{i : \max_{\boldsymbol{x} \in \mathsf{C}_i} l(\boldsymbol{x}) = \tau\}.$$

Notice that we only pick up terms when the hyperplane $\mathsf{H}_\tau$ descends into one of the convex bodies $\mathsf{C}_i$. We get no contribution when $\mathsf{H}_\tau$ leaves the bottom of $\mathsf{C}_i$. Indeed, when $\mathsf{C}_i \cap \mathsf{H}_\tau = \varnothing$ we must also have $\lim_{\varepsilon \downarrow 0} \mathsf{C}_i \cap \mathsf{H}_{\tau+\varepsilon} = \varnothing$. See Figure 10.3 for an illustration.

Since $l$ has a unique maximum value on each of the $m$ convex bodies $\mathsf{C}_i$, there are at most $m$ distinct points $\tau$ where $\chi(f_\tau)$ changes value. We now *define* $\chi(f)$ to be the sum of contributions from the discontinuities:

$$\chi(f) := \sum_{\tau \in \mathbb{R}} \Big[\chi(f_\tau) - \lim_{\varepsilon \downarrow 0} \chi(f_{\tau+\varepsilon})\Big]. \tag{10.4.2}$$

If $f = [\mathsf{C}]$ is the indicator of a nonempty convex body in $\mathscr{C}_d$, then $l$ has a unique maximum value on $\mathsf{C}$, so $\chi(\mathsf{C}) = 1$, as required.

We need to confirm that $\chi$, as defined in (10.4.2), is indeed a valuation. The restriction map is linear, so

$$(\alpha f + \beta g)_\tau = \alpha f_\tau + \beta f_\tau \quad \text{for } f, g \in \mathbb{A}(\mathscr{C}_d).$$

By the inductive hypothesis, $\chi$ is a linear valuation on the subalgebra $\{f_\tau : f \in \mathbb{A}(\mathscr{C}_d)\}$. That is,

$$\chi((\alpha f + \beta g)_\tau) = \alpha\chi(f_\tau) + \beta\chi(g_\tau).$$

By the linearity of the sum and limit in (10.4.2), we determine that

$$\chi(\alpha f + \beta g) = \alpha\chi(f) + \beta\chi(g).$$

We conclude that $\chi$ is a linear valuation on $\mathbb{A}(\mathscr{C}_d)$.

Finally, we note that the formula (10.4.1) holds if we specialize (10.4.2) to a function of the form $f = [\mathsf{A}]$. □

## 10.5 The Euler–Poincaré–Schläfli Formula

In this section, we will apply Hadwiger's construction of the Euler characteristic to prove the first theorem of algebraic topology.

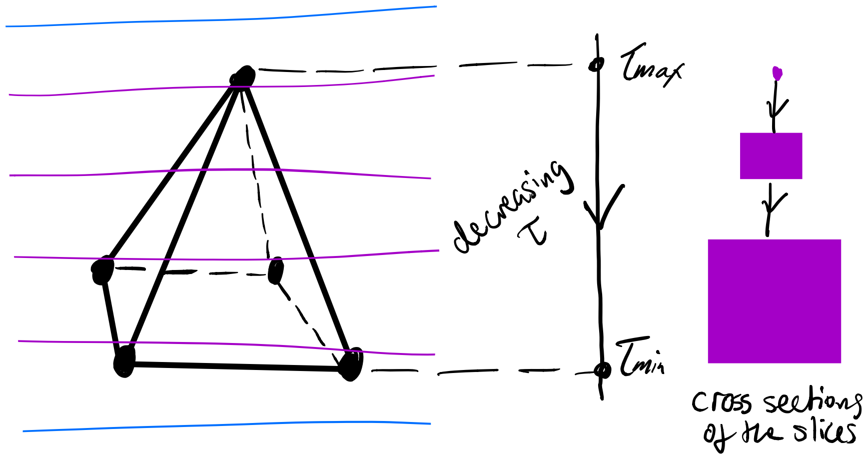


**Figure 10.4** (Hadwiger slices up a polytope). As a hyperplane scans through a polytope, the cross sections are also polytopes of lower dimension.

**Theorem 10.5.1** (Euler–Poincaré–Schläfli). *Let* $\mathsf{P}$ *be a polytope with* $\dim \mathsf{P} = d$*. Define* $f_i(\mathsf{P})$ *to be the number of faces of* $\mathsf{P}$ *with dimension* $i$*. Then*

$$\sum_{i=0}^{d} (-1)^i f_i(\mathsf{P}) = 1.$$

This famous formula is originally due to Euler in $\mathbb{R}^3$. It was extended to higher dimensions by Schläfli, and the first complete proof was given by Poincaré.

For instance, let us consider some of the low-dimensional cases:

$$\begin{aligned} \text{vertices} - \text{edges} &= 0 \quad \text{when } \dim \mathsf{P} = 2; \\ \text{vertices} - \text{edges} + \text{facets} &= 2 \quad \text{when } \dim \mathsf{P} = 3. \end{aligned}$$

Most of the work for proving the EPS formula will occur in the following lemma.

**Lemma 10.5.2** (Some Euler characteristics). *For a polytope* $\mathsf{P}$ *with* $\dim \mathsf{P} = d$*, we have*

$$\begin{aligned} \chi(\operatorname{bd} \mathsf{P}) &= 1 + (-1)^{d-1} \\ \chi(\operatorname{int} \mathsf{P}) &= (-1)^d \end{aligned}$$

*Proof.* As we have seen, every point in a convex set belongs to some face, and proper faces are contained in the boundary. Therefore, the boundary of the polytope may be decomposed as

$$\operatorname{bd} \mathsf{P} = \bigcup_{\mathsf{F} \lhd \mathsf{P},\, \mathsf{F} \neq \mathsf{P}} \mathsf{F}.$$

In particular, the indicator $[\operatorname{bd} \mathsf{P}]$ belongs to the algebra of polytopes $\mathbb{A}(\mathscr{P}_d)$, which means that $\chi(\operatorname{bd} \mathsf{P})$ is defined.

We may calculate $\chi(\operatorname{bd} \mathsf{P})$ by induction on the dimension of $\mathsf{P}$. When $d = 1$, the polytope $\mathsf{P}$ is an interval, so the boundary of $\mathsf{P}$ consists of two different points. Thus, the Euler characteristic $\chi(\operatorname{bd} \mathsf{P}) = 2 = 1 + (-1)^{1-1}$.

Assume that the result holds for dimension $d-1$, and consider a $d$-dimensional polytope $\mathsf{P}$. We will invoke the construction in Theorem 10.4.1. See Figure 10.4 for an illustration.

Let us consider slices of the polytope, i.e., the intersections $\mathsf{P} \cap \mathsf{H}_\tau$ with hyperplanes $\mathsf{H}_\tau = \{\boldsymbol{x} \in \mathbb{R}^d : x_d = \tau\}$. Define $\tau_{\max}$ and $\tau_{\min}$ to be the values of $\tau$ where the hyperplane $\mathsf{H}_\tau$ respectively enters and leaves the polytope $\mathsf{P}$ as we decrease $\tau$. Formally, $\tau_{\max} = \max_{\boldsymbol{x} \in \mathsf{P}} l(\boldsymbol{x})$, and $\tau_{\min} = \min_{\boldsymbol{x} \in \mathsf{P}} l(\boldsymbol{x})$, where $l$ reports the last coordinate of a vector. To compute the Euler characteristic of the intersection with $\mathsf{H}_\tau$, we have three cases to consider:

1. For $\tau_{\min} < \tau < \tau_{\max}$, the intersection $\mathsf{P} \cap \mathsf{H}_\tau$ is a $d-1$ dimensional polytope (see the homework!), and the boundary $\operatorname{bd}(\mathsf{P} \cap \mathsf{H}_\tau) = (\operatorname{bd} \mathsf{P}) \cap \mathsf{H}_\tau$. The inductive hypothesis shows that the Euler characteristic of the intersection satisfies

$$\chi((\operatorname{bd} \mathsf{P}) \cap \mathsf{H}_\tau) = \chi(\operatorname{bd}(\mathsf{P} \cap \mathsf{H}_\tau)) = 1 + (-1)^{d-2}.$$

2. For each $\tau \in \{\tau_{\min}, \tau_{max}\}$, the intersection $\mathsf{P} \cap \mathsf{H}_\tau$ is a nonempty face of $\mathsf{P}$ contained in $\operatorname{bd} \mathsf{P}$. Indeed, the intersection is the maximum of a linear functional on the set $\mathsf{P}$. As a consequence,

$$\chi((\operatorname{bd} \mathsf{P}) \cap \mathsf{H}_\tau) = \chi(\mathsf{P} \cap \mathsf{H}_\tau) = 1$$

because each face of a polytope is a polytope.

3. For $\tau < \tau_{\min}$ and $\tau > \tau_{\max}$, it is clear that

$$\chi((\operatorname{bd} \mathsf{P}) \cap \mathsf{H}_\tau) = \chi(\varnothing) = 0.$$

Armed with this information, we may now compute the Euler characteristic of the boundary $\operatorname{bd} \mathsf{P}$ using Theorem 10.4.1. Noticing that the only interesting things happen at the top and bottom, we get

$$\begin{aligned} \chi(\operatorname{bd} \mathsf{P}) &= \sum\nolimits_{\tau \in \mathbb{R}} \left[ \chi((\operatorname{bd} \mathsf{P}) \cap \mathsf{H}_\tau) - \lim_{\varepsilon \downarrow 0} \chi((\operatorname{bd} \mathsf{P}) \cap \mathsf{H}_{\tau+\varepsilon}) \right] \\ &= (1) + (1 - (1 + (-1)^{d+2})) \\ &= 1 + (-1)^{d-1}. \end{aligned}$$

The first contribution in the second line comes from $\tau_{\max}$, when the hyperplane enters the top of the polytope, and the second contribution comes from $\tau_{\min}$, when the hyperplane leaves the bottom of the polytope.

Finally, we compute the Euler characteristic of the the interior using the relation $\chi(\mathsf{P}) = \chi(\operatorname{bd} \mathsf{P}) + \chi(\operatorname{int} \mathsf{P})$. □

Using this lemma, we may swiftly prove the EPS formula.

*Proof of the EPS formula.* Recall that a polytope $\mathsf{P}$ admits a disjoint decomposition into faces. Therefore,

$$[\mathsf{P}] = \sum\nolimits_{\mathsf{F} \lhd \mathsf{P}} [\operatorname{relint} \mathsf{F}].$$

By linearity of the Euler characteristic and the previous lemma,

$$\begin{aligned} 1 = \chi(\mathsf{P}) &= \sum\nolimits_{\mathsf{F} \lhd \mathsf{P}} \chi(\operatorname{relint} \mathsf{F}) \\ &= \sum\nolimits_{\mathsf{F} \lhd \mathsf{P}} (-1)^{\dim \mathsf{F}} \\ &= \sum\nolimits_{i=0}^{d} (-1)^i f_i(\mathsf{P}). \end{aligned}$$

This endeth the lesson. □

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# Lecture 11: Integral Geometry

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*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*6 November 2018*

## 11.1 Agenda for Lecture 11

We begin with a famous geometric problem, called Buffon's Needle. The most common solution is based on conditional probability computations. We present an alternative solution using purely geometric arguments. This is one of the earliest result in the field of integral geometry, which concerns questions about averaging over geometric groups.

This lecture gives an introduction to integral geometry. Our approach is based on Hadwiger's functional theorem. This result states that every "nice" valuation on convex bodies can be expressed in terms of the intrinsic volumes. Hadwiger's work demonstrates that the intrinsic volumes are truly fundamental to convex geometry.

Using Hadwiger's functional theorem, we develop two general results in integral geometry. First, we establish Crofton's formula, which gives an expression for the total measure of the set of affine flats that touch a convex body. Second, we establish the principal kinematic formula, which gives an expression for the total measure of the set of rigid motions that bring two convex bodies into contact with each other.

1. Buffon's Needle
2. Hadwiger's Theorems
3. Grassmannians and Invariant Measures
4. Crofton's Formula
5. Principal Kinematic Formula

## 11.2 Buffon's Needle

To begin, we describe Buffon's needle problem, and we give a geometrical solution. Further discussion appears in [KR97, Sec. 1].

### 11.2.1 The Problem

Consider an infinite family of parallel lines in the plane with unit separation between adjacent pairs. Suppose that we toss a needle of length $L$ "randomly" onto the plane. What is the expected number of intersections between the needle and the lines? See Figure 11.1 for an illustration.

It is important to explain what we mean by "random" here. We assume that one endpoint of the needle is distributed uniformly over the space (hence both endpoints), and the orientation of the needle is also uniformly distributed. The following solution depends heavily on these assumptions.

### 11.2.2 Solution

We present an ingenious solution, originally proposed by Barbier in [Bar60]. Crofton gave a far-reaching extension of this method in [Cro68].

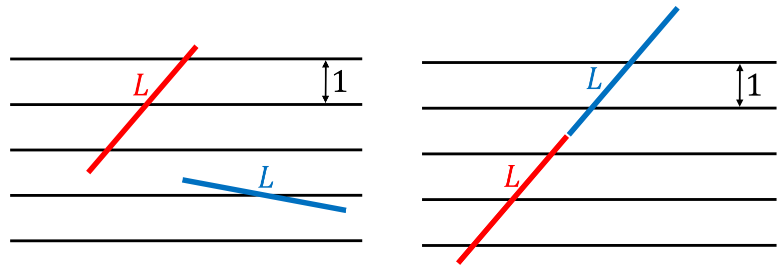


**Figure 11.1** (Buffon's Needle). A needle is tossed randomly on a grid of lines. [left] Due to the uniform distribution assumption, two needles of length $L$ can be assumed to be dependent on each other and [right] form a needle of length $2L$.

Let $X_L$ be a random variable defined as

$$X_L := \#\{\text{intersections between a random needle of length } L \text{ and the grid}\}.$$

Define a function

$$f(L) := \mathbb{E}[X_L] = \text{expected number of intersections}.$$

Let $X_L$ and $X'_L$ be two instances of the random variable. Then

$$2f(L) = \mathbb{E}[X_L] + \mathbb{E}[X'_L] = \mathbb{E}[X_L + X'_L].$$

The above equality holds true regardless of whether $X_L$ and $X'_L$ are statistically independent! Indeed, we can allow arbitrary dependency between $X_L$ and $X'_L$.

Suppose that two needles are welded together as illustrated in Figure 11.1[right]. The endpoint and the orientation of the first needle are uniformly distributed. The endpoint of the second needle is welded to the endpoint of the first needle, so it inherits the uniform distribution of its endpoint and orientation from the first needle. Therefore, even if the two needles are attached, $X_L$ and $X'_L$ have the same distribution. Moreover $X_L + X'_L$ has the same distribution as $X_{2L}$, the number of intersections of a needle of twice the length. Thus,

$$2f(L) = \mathbb{E}[X_L + X'_L] = \mathbb{E}[X_{2L}] = f(2L).$$

Repeating the same argument, we find that

$$qf(L) = f(qL) \quad \text{for } q \in \mathbb{Q}_+.$$

It is also clear that $f$ is an increasing function. Indeed, the longer the needle, the larger the expected number of intersections. Therefore, we can say that

$$\alpha f(L) = f(\alpha L) \quad \text{for } \alpha \in \mathbb{R}_+. \tag{11.2.1}$$

In other words, $f$ is positively homogeneous.

We can now think about different ways to concatenate needles. For example, consider Figure 11.2[left]. We can impose arbitrary dependency structure on several $X_{L_i}$ without changing the expectation:

$$\mathbb{E}[X_{L_1} + X_{L_2} + X_{L_3} + X_{L_4}] = f(L_1 + L_2 + L_3 + L_4).$$

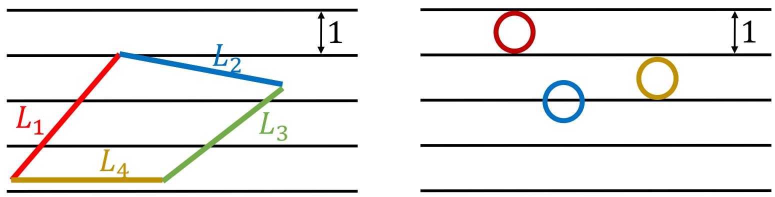


**Figure 11.2** (Different Configurations). [left] Due to the flexibility in the dependency, we can form an arbitrary shape from $n$ different needles. [right] Using limiting arguments, we can arrange infinitesimal needles to form a circle.

More generally,

$$\mathbb{E}\Big[\sum_{i=1}^{m} X_{L_i}\Big] = f\Big(\sum_{i=1}^{m} L_i\Big)$$

In summary, we can arrange needles of arbitrary lengths into any configuration. The expected number of intersections is always proportional to the total length.

We now consider a circular ring with circumference $L$. Define

$$C_L = \#\{\text{intersections between a random ring with circumference } L \text{ and the grid}\}.$$

By taking limits, we can think about the ring as being composed of infinitesimal needles. The previous argument shows that

$$\mathbb{E}[C_L] = f(L).$$

In particular, consider the ring with diameter one and circumference $\pi$. The number of intersections between this ring and the grid is no longer random: it always equals two. See Figure 11.2[right]. Therefore,

$$2 = \mathbb{E}[C_\pi] = f(\pi).$$

For any $L > 0$, the positive homogeneity relation (11.2.1) implies that

$$2L = f(\pi L) = \pi f(L).$$

We conclude that

$$f(L) = \frac{2L}{\pi}.$$

This is the solution to Buffon's needle problem.

Remark 11.2.1 (The random model). Notice that this argument really requires the uniform distribution assumption. Otherwise, the endpoint of $X_L$ and its orientation may not be a valid starting point for $X'_L$, in which case the concatenation of $X_L$ and $X'_L$ may not be an instance of $X_{2L}$. This problem persists for more complicated configurations.

### 11.2.3 Integral Geometry

Buffon's needle problem involves volumetric properties (i.e., nonempty intersection) of geometric objects (i.e., needles and grids). It also involves averaging over a geometric group (i.e., the set of rigid motions).

The field of *integral geometry*, or *geometric probability*, concerns this general class of problems. This lecture will develop some important results from integral geometry. The tools, however, are much more sophisticated than the simple arguments we applied above.

## 11.3 Hadwiger's Theorems

The most direct approach to integral geometry requires some deep theorems of Hadwiger [Had57] that characterize valuations on convex bodies. This section introduces Hadwiger's results without proof. We begin with a definition.

**Definition 11.3.1** (Simple valuation). A set valuation $\mu$ on $\mathscr{C}_d$ is *simple* if $\dim \mathsf{C} < d$ implies that $\mu(\mathsf{C}) = 0$.

**Example 11.3.2** (Volume). The volume $\mathrm{Vol}_d$ on $\mathscr{C}_d$, derived from the Lebesgue measure on $\mathbb{R}^d$, is a simple valuation.

Hadwiger's characterization theorem shows that volume is the only simple set valuation that is compatible with Euclidean geometry.

**Theorem 11.3.3** (Hadwiger's Characterization). *Let $\mu$ be a simple, Hausdorff continuous valuation on $\mathscr{C}_d$ that is invariant under proper rigid motions (i.e., translation and rotation). Then*

$$\mu = \mathrm{const} \cdot \mathrm{Vol}_d .$$

*Proof.* See [KR97, Thm. 8.3.2] for a proof devised by Klain; this is considered to be the most direct approach. Hadwiger's book [Had57] contains the original proof, which is more difficult. □

In its spirit, Theorem 11.3.3 is similar to the fact that, up to scaling, Lebesgue measure is the *unique* translation invariant, additive, positive measure on $\mathbb{R}^d$. It is much easier to establish the classic result about Lebesgue measure because the measure must be additive on the entire class of Borel sets, which is enormous. In contrast, we are constraining the set valuation only on convex bodies, which are less numerous. Nevertheless, the assumptions in Theorem 11.3.3 are still strong enough to force the set valuation to coincide with the Lebesgue measure. The proof is quite difficult; it involves results about the dissection of convex bodies and representation theory.

Hadwiger's characterization theorem has an immediate and spectacular consequence: The intrinsic volumes are essentially the only set valuations that are compatible with Euclidean geometry.

**Corollary 11.3.4** (Hadwiger's Functional Theorem). *Let $\mu$ be a Hausdorff continuous valuation on $\mathscr{C}_d$ that is invariant under proper rigid motions. Then $\mu$ is a linear combination of intrinsic volumes $V_j$:*

$$\mu = \sum\nolimits_{j=0}^{d} \alpha_j V_j \quad \text{for } \alpha_j \in \mathbb{R}.$$

*Proof.* This result follows from an easy induction argument using Hadwiger's characterization theorem. For details, see [KR97, Thm. 9.1.1]. □

Hadwiger's functional theorem reveals that the intrinsic volumes are truly fundamental quantities in Euclidean geometry. In the rest of this lecture, we will see how this fact leads to striking results in integral geometry.

## 11.4 Grassmannians and Invariant Measures

Before we continue with our geometric questions, we need to spend a few minutes to understand how we compute volumes of some geometric sets.

First, recall that the *Grassmannian* $\mathscr{G}(j,d)$ is the family of all $j$-dimensional subspaces in $\mathbb{R}^d$:

$$\mathscr{G}(j,d) := \left\{ \mathsf{L} \subset \mathbb{R}^d : \mathsf{L} \text{ is a } j\text{-dimensional subspace} \right\}.$$

It is also natural to parameterize the Grassmannian in terms of rotations of a given subspace. We write $\mathsf{SO}(d)$ for the group of $d \times d$ rotation matrices; that is, the $d \times d$ orthogonal matrices with determinant one. Let $\mathsf{L}_0 \in G(j,d)$ be a fixed $j$-dimensional subspace. Then

$$\mathscr{G}(j,d) = \{\boldsymbol{U}\mathsf{L}_0 : \boldsymbol{U} \in \mathsf{SO}(d)\}.$$

So the Grassmannian is the orbit of a single subspace under the rotation group. Note, however, that this orbit covers the Grassmannian many times.

We are going to construct a probability measure $\nu_j$ on the Grassmannian $\mathscr{G}(j,d)$ that is invariant under rotations. First, recall that there is a rotation-invariant probability measure $\nu$ on the group $\mathsf{SO}(d)$ of rotation matrices. Draw a random orthogonal matrix $\boldsymbol{Q} \in \mathsf{SO}(d)$ according to the measure $\nu$. Then $\boldsymbol{Q}\mathsf{L}_0$ is a uniformly random element of the Grassmannian $\mathscr{G}(j,d)$. Up to scaling, $\nu_j$ is the push-forward of $\nu$ onto $\mathscr{G}(j,d)$ via the map $\boldsymbol{Q} \mapsto \boldsymbol{Q}\mathsf{L}_0$. We normalize $\nu_j$ so that the total measure of the $\mathscr{G}(j,d)$ is one.

Next, we introduce the *affine Grassmannian*:

$$\mathscr{A}(j,d) = \left\{ \mathsf{A} \subset \mathbb{R}^d : \mathsf{A} \text{ is a } j\text{-dimensional affine set} \right\}.$$

Each affine set $\mathsf{A} \in \mathscr{A}(j,d)$ can be expressed uniquely as a translation of a subspace:

$$\mathsf{A} = \mathsf{L} + \boldsymbol{x} \quad \text{where} \quad \mathsf{L} \in \mathscr{G}(j,d) \quad \text{and} \quad \boldsymbol{x} \in \mathsf{L}^\perp.$$

This representation suggests how we can construct a measure $\mu_j$ on the affine Grassmannian $\mathscr{A}(j,d)$ that is invariant under rigid motions. First, we select a subspace $\mathsf{L} \in \mathscr{G}(j,d)$ from the uniform distribution $\nu_j$. The translation $\boldsymbol{x} \in \mathsf{L}^\perp$ follows the Lebesgue measure on $\mathsf{L}^\perp$.

It is sometimes more convenient to express an affine set $\mathsf{A} \in \mathscr{A}(j,d)$ in terms of a fixed subspace $\mathsf{L}_0 \in \mathscr{G}(j,d)$:

$$\mathsf{A} = \boldsymbol{U}(\mathsf{L}_0 + \boldsymbol{x}) \quad \text{where} \quad \boldsymbol{x} \in \mathsf{L}_0^\perp \quad \text{and} \quad \boldsymbol{U} \in \mathsf{SO}(d).$$

Up to scaling, the measure $\mu_j$ on the affine Grassmannian is the push-forward of the product measure $\mathrm{Leb}(\mathsf{L}_0^\perp) \times \nu$ via the map $(\boldsymbol{x}, \boldsymbol{Q}) \mapsto \boldsymbol{Q}(\mathsf{L}_0 + \boldsymbol{x})$. We have normalized the measure $\mu_j$ so that

$$\mu_j\{\mathsf{A} \in \mathscr{A}(j,d) : \mathsf{A} \cap \mathsf{B}_d \neq \emptyset\} = \kappa_{d-j}.$$

In other words, the total measure of the set of $j$-dimensional flats that touch the ball $\mathsf{B}_d$ is the volume of the ball $\mathsf{B}_{d-j}$.

## 11.5 Crofton's Formula

We are now prepared to study another geometric question: What is the measure of the set of affine flats of dimension $j$ that touch a convex body in $\mathbb{R}^d$?

Let $\mathsf{C} \in \mathscr{C}_d$ be a convex body. The measure of set of flats of dimension $j$ that touch the convex body $\mathsf{C}$ can be written as

$$\underbrace{\mu_j\{\mathsf{A} \in \mathscr{A}(j,d) : \mathsf{C} \cap \mathsf{A} \neq \emptyset\}}_{\text{Total measure of flats that touch } \mathsf{C}} = \int_{\mathscr{A}(j,d)} \underbrace{\chi(\mathsf{C} \cap \mathsf{A})}_{\substack{\text{Euler} \\ \text{characteristic}}} \quad \underbrace{\mathrm{d}\mu_j(\mathsf{A})}_{\substack{\text{integrate over} \\ \text{subspace and translate}}}$$

The Euler characteristic returns one precisely when the flat $\mathsf{A}$ and the convex body $\mathsf{C}$ have a nontrivial intersection.

Crofton's formula gives an answer to the question about the measure of the set of affine flats that touch a convex body. It also provides a new interpretation of the intrinsic volumes.

**Theorem 11.5.1** (Crofton's formula). *Let $\mathsf{C} \in \mathscr{C}_d$ be a nonempty convex body. Then*

$$\int_{\mathscr{A}(j,d)} \chi(\mathsf{C} \cap \mathsf{A})\, \mathrm{d}\mu_j(\mathsf{A}) = \begin{bmatrix} d \\ j \end{bmatrix}^{-1} V_{d-j}(\mathsf{C})$$

*where the* flag coefficient *is defined as*

$$\begin{bmatrix} d \\ j \end{bmatrix} := \binom{d}{j} \frac{\kappa_d}{\kappa_j \kappa_{d-j}}.$$

When we apply Crofton's formula for specific choices of the dimension $j$ of the flat, we obtain interpretations of specific intrinsic volumes. For example,

- **Volume ($j = 0$).** The volume $V_d$ is the measure of the set of points that touch $\mathsf{C}$.
- **Surface area ($j = 1$).** The surface area $V_{d-1}$ is twice the measure of the set of affine lines that pierce $\mathsf{C}$.
- **Mean width ($j = d - 1$).** The mean width $V_1$ is proportional to the measure of the set of affine hyperplanes that intersect $\mathsf{C}$.
- **Euler characteristic ($j = d$).** The Euler characteristic $V_0$ is equal to one because $\mathbb{R}^d$ intersects the nonempty set $\mathsf{C}$.

Let us establish the result.

*Proof.* Define a functional $\varphi : \mathscr{C}_d \to \mathbb{R}$ via the rule

$$\varphi(\mathsf{C}) = \int_{\mathscr{A}(j,d)} \chi(\mathsf{C} \cap \mathsf{A})\, \mathrm{d}\mu_j(\mathsf{A}).$$

Since $\chi$ is a set valuation and the integral is linear, it follows that $\varphi$ is a set valuation. Since $\chi$ is continuous, it follows that $\varphi$ is continuous. Let $\boldsymbol{T}$ be a rigid motion. Since $\chi$ and $\mu_j$ are invariant under rigid motions,

$$\begin{aligned} \varphi(\boldsymbol{T}\mathsf{C}) &= \int_{\mathscr{A}(j,d)} \chi((\boldsymbol{T}\mathsf{C}) \cap \mathsf{A})\, \mathrm{d}\mu_j(\mathsf{A}) \\ &= \int_{\mathscr{A}(j,d)} \chi(\mathsf{C} \cap (\boldsymbol{T}^{-1}\mathsf{A}))\, \mathrm{d}\mu_j(\mathsf{A}) \\ &= \int_{\mathscr{A}(j,d)} \chi(\mathsf{C} \cap \mathsf{A}))\, \mathrm{d}\mu_j(\mathsf{A}) = \varphi(\mathsf{C}). \end{aligned}$$

We see that $\varphi$ is a continuous set valuation that is invariant under rigid motions. Hadwiger's functional theorem now implies that

$$\varphi(\mathsf{C}) = \sum\nolimits_{j=0}^{d} \alpha_j V_j(\mathsf{C})$$

for some coefficients $\alpha_j \in \mathbb{R}$. We must determine the coefficients.

First, we claim that $\varphi$ is homogeneous of degree $d-j$. In that case, because the intrinsic volume $V_i$ is homogeneous of degree $i$, we must have

$$\varphi(\mathsf{C}) = \alpha_{d-j} \cdot V_{d-j}(\mathsf{C}). \tag{11.5.1}$$

To verify homogeneity, select $\lambda > 0$. For a subspace $\mathsf{L}$, we have the relations

$$\lambda\mathsf{C} \cap (\mathsf{L} + \boldsymbol{x}) \neq \emptyset \quad \text{if and only if} \quad \mathsf{C} \cap (\mathsf{L} + \lambda^{-1}\boldsymbol{x}) \neq \emptyset.$$

Therefore, if we write $\mathsf{A} = \mathsf{L} + \boldsymbol{x}$ and use Fubini's theorem to factor the integral over $\mu_j$,

$$\begin{aligned}
\varphi(\lambda\mathsf{C}) &= \int_{\mathscr{G}(j,d)} \mathrm{d}\nu_j(\mathsf{L}) \int_{\mathsf{L}^\perp} \mathrm{d}\boldsymbol{x}\, \chi(\lambda\mathsf{C} \cap (\mathsf{L} + \boldsymbol{x})) \\
&= \int_{\mathscr{G}(j,d)} \mathrm{d}\nu_j(\mathsf{L}) \int_{\mathsf{L}^\perp} \mathrm{d}\boldsymbol{x}\, \chi(\mathsf{C} \cap (\mathsf{L} + \lambda^{-1}\boldsymbol{x})) \\
&= \lambda^{d-j} \int_{\mathscr{G}(j,d)} \mathrm{d}\nu_j(\mathsf{L}) \int_{\mathsf{L}^\perp} \mathrm{d}\boldsymbol{x}\, \chi(\mathsf{C} \cap (\mathsf{L} + \boldsymbol{x})) = \lambda^{d-j}\varphi(\mathsf{C}).
\end{aligned}$$

The factor $\lambda^{d-j}$ comes from the Jacobian of the change of variables $\lambda^{-1}\boldsymbol{x} \mapsto \boldsymbol{x}$ on the $(d-j)$-dimensional space $\mathsf{L}^\perp$.

In the expression (11.5.1), the constant $\alpha_{d-j}$ does not depend on $\mathsf{C}$. Therefore, we can determine its value by choosing a suitable set $\mathsf{C}$. The natural choice is $\mathsf{C} = \mathsf{B}_d$. From Q3(a) of Homework #2, we know intrinsic volumes of the unit ball:

$$V_{d-j}(\mathsf{B}_d) = \binom{d}{j} \frac{\kappa_d}{\kappa_j}.$$

In order to compute $\varphi(\mathsf{B}_d)$, fix a subspace $\mathsf{L}_0$. Notice that

$$\chi(\mathsf{B}_d \cap (\mathsf{L}_0 + \boldsymbol{x})) = 1 \quad \text{if and only if} \quad \boldsymbol{x} \in \mathsf{L}_0^\perp \cap \mathsf{B}_d.$$

Write $\mathsf{A} = \boldsymbol{Q}(\mathsf{L}_0 + \boldsymbol{x})$ for $\boldsymbol{x} \in \mathsf{L}_0^\perp$, and factor the integral to obtain

$$\begin{aligned}
\varphi(\mathsf{B}_d) &= \int_{\mathsf{SO}(d)} \mathrm{d}\nu(\boldsymbol{Q}) \int_{\mathsf{L}_0^\perp} \chi(\mathsf{B}_d \cap \boldsymbol{Q}(\mathsf{L}_0 + \boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathsf{SO}(d)} \mathrm{d}\nu(\boldsymbol{Q}) \int_{\mathsf{L}_0^\perp \cap \mathsf{B}_d} \mathrm{d}\boldsymbol{x} = \kappa_{d-j}.
\end{aligned}$$

For $\mathsf{C} = \mathsf{B}_d$, the formula (11.5.1) reduces to

$$\kappa_{d-j} = \alpha_{d-j} \cdot \binom{d}{j} \frac{\kappa_d}{\kappa_j}.$$

This relation determines $\alpha_{d-j}$. Combine with (11.5.1) to complete the argument. □

Remark 11.5.2 (Euler characteristic). There is nothing special about the appearance of the Euler characteristic in this argument. We can replace it with other intrinsic volumes and perform similar calculations.

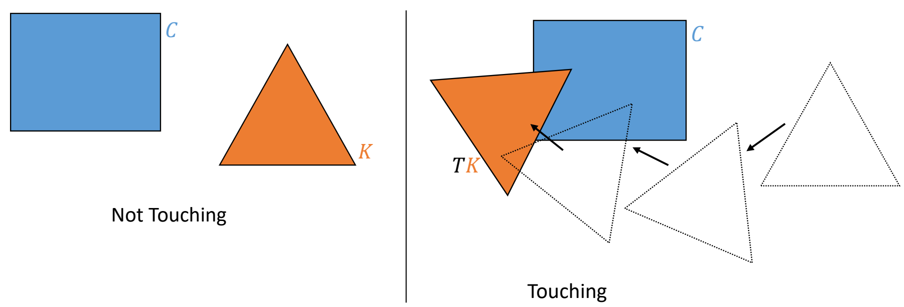


**Figure 11.3** (Rigid Motions). The rigid motion $\boldsymbol{T}$ translates and rotates the convex body $\mathsf{K}$ so that $\mathsf{C}$ and $\boldsymbol{T}\mathsf{K}$ have a nonempty intersection.

## 11.6 Principal Kinematic Formula

Finally, we will consider another question in integral geometry: What is the measure of the set of rigid motions that bring two convex bodies into contact with each other? See Figure 11.3 for an illustration.

The family $\mathscr{E}_d$ of rigid motions on $\mathbb{R}^d$ can be parameterized as

$$\mathscr{E}_d := \{(\boldsymbol{x}, \boldsymbol{Q}) : \boldsymbol{x} \in \mathbb{R}^d \text{ and } \boldsymbol{Q} \in \mathsf{SO}(d)\}.$$

A pair $\boldsymbol{T} = (\boldsymbol{x}, \boldsymbol{Q}) \in \mathscr{E}_d$ acts via the rule.

$$\boldsymbol{T}\boldsymbol{z} := \boldsymbol{Q}\boldsymbol{z} + \boldsymbol{x}.$$

We equip the set $\mathscr{E}_d$ with the product of the Lebesgue measure $\mathrm{Vol}_d$ on the first coordinate and the rotation-invariant measure $\nu$ of the second coordinate.

Our goal is to compute

$$\mu\{\boldsymbol{T} \in \mathscr{E}_d : \mathsf{C} \cap \boldsymbol{T}\mathsf{K} \neq \emptyset\} = \int_{\mathscr{E}_d} \chi(\mathsf{C} \cap \boldsymbol{T}\mathsf{K}) \, \mathrm{d}\mu(\boldsymbol{T}).$$

We have the following amazing result.

**Theorem 11.6.1** (Principal kinematic formula). *Let $\mathsf{C}, \mathsf{K} \in \mathscr{C}_d$ be nonempty convex bodies. Then*

$$\int_{\mathscr{E}_d} \chi(\mathsf{C} \cap \boldsymbol{T}\mathsf{K}) \, \mathrm{d}\mu(\boldsymbol{T}) = \sum\nolimits_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}^{-1} V_j(\mathsf{C}) V_{d-j}(\mathsf{K}).$$

Some remarks are in order:

- Theorem 11.6.1 is referred to as a "kinematic formula" because it concerns moving convex bodies.
- The kinematic formula is not obvious in the plane, let alone in higher-dimensional Euclidean spaces.
- Another suprising fact is that the formula only involves the intrinsic volumes of the individual convex bodies $\mathsf{C}$ and $\mathsf{K}$; no "joint" characteristics of the pair $(\mathsf{C}, \mathsf{K})$ appear.

- There is nothing special about the choice of the Euler characteristic in this formula. We can also develop "nonprincipal" kinematic formulas for other intrinsic volumes.
- Using Groemer's extension theorem, we can even lift kinematic formula to the algebra of convex bodies.

We will now provide a very rough sketch for the proof of the kinematic formula.

*Proof.* Here are the basic steps:

1. Define a functional $\varphi(\mathsf{C}, \mathsf{K})$ via the rule

$$\varphi(\mathsf{C}, \mathsf{K}) = \int_{\mathscr{E}_d} \chi(\mathsf{C} \cap \boldsymbol{T}\mathsf{K}) \, \mathrm{d}\mu(\boldsymbol{T}).$$

2. Show that $\varphi$ is symmetric in its coordinates.
3. Show that $\varphi(\mathsf{C}, \cdot)$ is a "nice" valuation.
4. Apply Hadwiger's functional theorem to obtain

$$\varphi(\mathsf{C}, \cdot) = \sum\nolimits_{j=0}^{d} \alpha_j(\mathsf{C}) V_j(\cdot).$$

5. Prove that each $\alpha_j(\cdot)$ is a "nice" valuation.
6. Apply Hadwiger's functional theorem to each of the $\alpha_j(\cdot)$ to obtain

$$\alpha_j(\mathsf{C}) = \sum\nolimits_{i=0}^{d} \alpha_{ij} V_i(\mathsf{C}).$$

7. Combine these results to see that

$$\varphi(\mathsf{C}, \mathsf{K}) = \sum\nolimits_{i,j=0^d} \alpha_{ij} V_i(\mathsf{C}) V_j(\mathsf{K}).$$

8. Choose $\mathsf{C} = \lambda \mathsf{B}_d$ and $\mathsf{K} = \mu \mathsf{B}_d$ to determine the constants $\alpha_{ij}$.

See [KR97, Thm. 10.1.1] or [Gru07, Sec. 7.4] for a complete proof. □

# Lecture 12: The Isoperimetric Problem

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*ACM 204, Fall 2018*
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*8 November 2018*

## 12.1 Agenda for Lecture 12

In this lecture, we will discuss the isoperimetric problem, a solution of the isoperimetric problem via the Brunn–Minkowski inequality, and a proof of the Brunn–Minkowksi inequality using the Prékopa–Leindler inequality. Last, we will prove the Prékopa–Leindler inequality.

1. Dido's problem and the isoperimetric theorem
2. Minkowski surface area
3. Isoperimetry and the Brunn–Minkowski inequality
4. Geometry and analytic inequalities

## 12.2 Dido's Problem

We can trace the origins of the isoperimetric problem to antiquity. Dido was a Phoenician princess who fled her home after her brother, Pygmalion of Tyre, assassinated her wealthy husband, Sychaeus. She arrived in northern Africa, in modern-day Libya. She asked the local ruler, King Iarbas, for a small plot of land—whatever she could enclose with the hide of a bull. He agreed. She cut the hide into a narrow strip and formed a large circle, enclosing the base of a hill. Thanks to Dido's skill at geometric thinking and *maroquinerie*, this hill became the city of Carthage.

We can pose the same question in a more abstract form. In the plane, among all closed curves of fixed length, which one(s) enclose the greatest area? Dido intuitively understood that the (only) answer is a circle. This is called the *isoperimetric problem* because it concerns curves of equal perimeter.

Dually, we can ask what is the minimum perimeter curve that encloses a fixed area. In three dimensions, the analogous problem requests the object with minimum surface area that encloses a fixed volume. Again, the unique solution is a Euclidean ball. This type of isoperimetric problem arises when we study soap bubbles. A single bubble is spherical because it chooses the configuration that minimizes the total surface energy, which is proportional to its surface area.

In this lecture, we consider a somewhat more general version of the isoperimetric problem. In $\mathbb{R}^d$, among all measurable sets with fixed surface area, which one(s) have the greatest volume? Equivalently, among all measurable sets with fixed volume, which one(s) have the least surface area? Today, we will prove the following version of the isoperimetric theorem.

**Theorem 12.2.1** (Isoperimetric theorem)**.** *In $\mathbb{R}^d$, among all convex bodies of fixed volume, a scaled Euclidean ball has the minimum surface area.*

In fact, the Euclidean ball is the *unique* convex body of minimum surface area. We will discuss this point later, but we omit the complete proof.

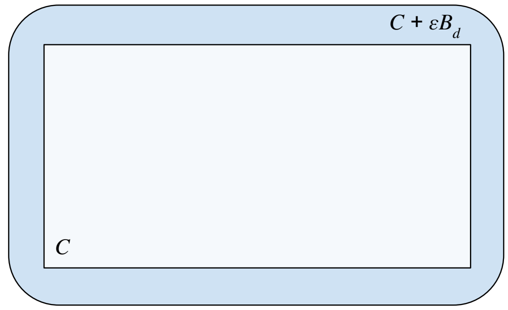


**Figure 12.1** (Minkowski surface area). The Minkowski surface area of $\mathsf{C}$ is the limiting difference between the volume of the parallel body, $\mathsf{C} + \varepsilon \mathsf{B}_d$ (blue) and the volume of $\mathsf{C}$ (light blue), relative to the thickness $\varepsilon$ of the shell as the shell becomes narrower.

The isoperimetric theorem actually holds for *all* measurable sets. We will prove this result—but only for a special notion of surface area. Unfortunately, a fully general treatment requires sophisticated ideas from the field of geometric measure theory.

## 12.3 Minkowski Surface Area

Before we can continue with our discussion of the isoperimetric theorem, we need to develop a notion of surface area for convex sets. The *Minkowski surface area* of a convex body is the volume of an infinitesimal shell surrounding the body, divided by the thickness of the shell. We obtain this shell as the difference between the body and an infinitesimal parallel body.

**Definition 12.3.1** (Minkowski surface area: Convex bodies)**.** Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty convex body. The *Minkowski surface area* of $\mathsf{C}$ is

$$S_{d-1}(\mathsf{C}) := \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\mathsf{C} + \varepsilon \mathsf{B}_d) - \mathrm{Vol}_d(\mathsf{C})}{\varepsilon}. \tag{12.3.1}$$

The limit in this expression exists, courtesy of Steiner's formula (see below). By changing variables, it is easy to see $S_{d-1}$ is homogeneous of degree $d-1$.

We have already encountered the Minkowski surface area in another context.

**Corollary 12.3.2** (Surface area and intrinsic volumes)**.** *Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty convex body. Then $S_{d-1}(\mathsf{C}) = 2\, V_{d-1}(\mathsf{C})$.*

*Proof.* By Steiner's formula and a change of index,

$$\begin{aligned} S_{d-1}(\mathsf{C}) &= \lim_{\varepsilon \downarrow 0} \frac{\sum_{j=0}^{d} \varepsilon^{d-j} \kappa_{d-j} V_j(\mathsf{C}) - \mathrm{Vol}_d(\mathsf{C})}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{\sum_{j=0}^{d} \varepsilon^{j} \kappa_{j} V_{d-j}(\mathsf{C}) - \varepsilon^0 \kappa_0\, V_d(\mathsf{C})}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \sum\nolimits_{j=1}^{d} \varepsilon^{j-1} \kappa_j V_{d-j}(\mathsf{C}) = \kappa_1\, V_{d-1}(\mathsf{C}). \end{aligned}$$

Since $\kappa_1 = \mathrm{Vol}_1(\mathsf{B}_1) = 2$, the statement follows. $\square$

For convex bodies, the Minkowski surface area agrees with other notions of surface area derived using measure theory or differential geometry. We can also extend the concept of Minkowski surface area to measurable sets, with some additional care.

**Definition 12.3.3** (Minkowski surface area: Measurable sets). Let $\mathsf{A} \subset \mathbb{R}^d$ be a measurable set. The (lower outer) *Minkowski surface area* of $\mathsf{A}$ is

$$S_{d-1}(\mathsf{A}) := \liminf_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\mathsf{A} + \varepsilon \mathsf{B}_d) - \mathrm{Vol}_d(\mathsf{A})}{\varepsilon}.$$

The Minkowski surface area coincides with other notions of surface area for sets that have piecewise smooth boundaries. For uglier sets, however, different flavors of surface area can give divergent results.

## 12.4 Isoperimetry and the Brunn-Minkowski Inequality

We will prove the isoperimetric theorem as a consequence of an important geometric inequality, associated with the names Brunn and Minkowski.

### 12.4.1 The Brunn–Minkowsi Inequality

The Brunn–Minkowski inequality states that the volume is a log-concave measure. It is one of the most fundamental facts in the entire field of geometry.

**Theorem 12.4.1** (Multiplicative Brunn-Minkowski inequality). *Let* $\mathsf{A}, \mathsf{E} \subset \mathbb{R}^d$ *be measurable sets. For* $\lambda \in [0, 1]$,

$$\mathrm{Vol}_d((1-\lambda)\mathsf{A} + \lambda \mathsf{E}) \geq \mathrm{Vol}_d(\mathsf{A})^{1-\lambda} \cdot \mathrm{Vol}_d(\mathsf{E})^{\lambda}. \tag{12.4.1}$$

*For convex bodies, equality holds if and only if*

1. $\mathsf{A}$ *or* $\mathsf{E}$ *is a single point;*
2. $\mathsf{A}$ *and* $\mathsf{E}$ *are contained in a parallel hyperplanes; or*
3. $\mathsf{A}$ *and* $\mathsf{E}$ *have nonempty interiors and are* homothetic*; that is, one set is a translation and/or dilation of the other.*

*In general, equality can hold only if the sets are convex with subsets of measure zero removed.*

In the next section, we will prove the Brunn–Minkowski inequality (omitting the equality condition).

### 12.4.2 Brunn–Minkowski implies the Isoperimetric Theorem

First, let us see how this result is relevant to the current discussion. The Brunn–Minkowski inequality quickly implies the isoperimetric theorem.

**Corollary 12.4.2** (Isoperimetric Inequality). *Let* $\mathsf{A} \subset \mathbb{R}^d$ *be a nonempty, measurable set. Then*

$$\left( \frac{S_{d-1}(\mathsf{A})}{S_{d-1}(\mathsf{B}_d)} \right)^{\frac{1}{d-1}} \geq \left( \frac{\mathrm{Vol}_d(\mathsf{A})}{\mathrm{Vol}_d(\mathsf{B}_d)} \right)^{\frac{1}{d}}. \tag{12.4.2}$$

*Equality holds if and only if* $\mathsf{A}$ *(up to a set of measure zero) is homothetic to a Euclidean ball.*

Let us note two particular consequences:

1. If $S_{d-1}(\mathsf{A}) = S_{d-1}(\mathsf{B}_d)$, then $\mathrm{Vol}_d(\mathsf{A}) \le \mathrm{Vol}_d(\mathsf{B}_d)$. For all sets with fixed Minkowski surface area, a scaled Euclidean ball has the maximum volume.
2. If $\mathrm{Vol}_d(\mathsf{A}) = \mathrm{Vol}_d(\mathsf{B}_d)$, then $S_{d-1}(\mathsf{A}) \ge S_{d-1}(\mathsf{B}_d)$. For all sets with fixed volume, a scaled Euclidean ball has the minimum Minkowski surface area.

Therefore, the isoperimetric inequality implies the isoperimetric theorem. The equality case establishes that the ball is the unique extremal solution.

*Proof.* Since the volume is homogeneous of degree $d$ and the Minkowski surface area is homogeneous of degree $d-1$, we may as well assume that $\mathrm{Vol}_d(\mathsf{A}) = \mathrm{Vol}_d(\mathsf{B}_d)$.

For $\varepsilon > 0$, we make the change of variables $\varepsilon = \lambda/(1-\lambda)$. The Brunn–Minkowksi inequality, Theorem 12.4.1, implies that

$$\begin{aligned}
S_{d-1}(\mathsf{A}) &= \liminf_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\mathsf{A} + \varepsilon \mathsf{B}_d) - \mathrm{Vol}_d(\mathsf{A})}{\varepsilon} \\
&= \liminf_{\lambda \downarrow 0} \frac{\mathrm{Vol}_d(((1-\lambda)\mathsf{A} + \lambda \mathsf{B}_d)/(1-\lambda)) - \mathrm{Vol}_d(\mathsf{A})}{\lambda/(1-\lambda)} \\
&= \liminf_{\lambda \downarrow 0} \frac{(1-\lambda)^{-d}\,\mathrm{Vol}_d((1-\lambda)\mathsf{A} + \lambda \mathsf{B}_d) - \mathrm{Vol}_d(\mathsf{A})}{\lambda/(1-\lambda)} \\
&\ge \liminf_{\lambda \downarrow 0} \frac{(1-\lambda)^{-d}\,\mathrm{Vol}_d(\mathsf{A})^{1-\lambda} \cdot \mathrm{Vol}_d(\mathsf{B}_d)^{\lambda} - \mathrm{Vol}_d(\mathsf{A})}{\lambda/(1-\lambda)} \\
&= \liminf_{\lambda \downarrow 0} \frac{(1-\lambda)^{-d}\,\mathrm{Vol}_d(\mathsf{B}_d) - \mathrm{Vol}_d(\mathsf{B}_d)}{\lambda/(1-\lambda)} \\
&= \liminf_{\lambda \downarrow 0} \frac{(1-\lambda)^{-d}\,\mathrm{Vol}_d((1-\lambda)\mathsf{B}_d + \lambda \mathsf{B}_d) - \mathrm{Vol}_d(\mathsf{B}_d)}{\lambda/(1-\lambda)} \\
&= \liminf_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\mathsf{B}_d + \varepsilon \mathsf{B}_d) - \mathrm{Vol}_d(\mathsf{B}_d)}{\varepsilon} = S_{d-1}(\mathsf{B}_d).
\end{aligned}$$

We have also used the fact that the volume is homogeneous of degree $d$. Finally, note that the inequality holds strictly unless $\mathsf{A}$ is homothetic to $\mathsf{B}_d$. □

## 12.5 Geometry and Analytic Inequalities

There are many proofs of the Brunn–Minkowski inequality. Today, we will pursue an approach that was pioneered by Liusternik in the 1930s. He realized that the Brunn–Minkowski inequality, Theorem 12.4.1, can be interpreted as a functional inequality applied to the indicator functions of the sets. By replacing sets with functions, we gain access to tools from analysis, which facilitates the proof. Moreover, this point of view shows that the Brunn–Minkowski inequality does not depend on the sets being convex.

### 12.5.1 The Prékopa–Leindler Inequality

The modern version of this approach proceeds via an inequality, independently developed by Prékopa and Leindler around 1970.

**Theorem 12.5.1** (Prékopa–Leindler inequality). *Fix $\lambda \in [0,1]$. Let $f, g, h : \mathbb{R}^d \to \mathbb{R}_+$ be nonnegative integrable functions that satisfy the inequality*

$$h((1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y}) \geq f(\boldsymbol{x})^{1-\lambda} g(\boldsymbol{y})^{\lambda} \quad \textit{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d. \tag{12.5.1}$$

*Then*

$$\int_{\mathbb{R}^d} h(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \geq \left( \int_{\mathbb{R}^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{1-\lambda} \left( \int_{\mathbb{R}^d} g(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\lambda}. \tag{12.5.2}$$

At first sight, this result looks perplexing. If anything, it may remind the reader of Hölder's inequality. For nonnegative integrable functions $f, g : \mathbb{R}^d \to \mathbb{R}_+$, set

$$k(\boldsymbol{x}) := f(\boldsymbol{x})^{1-\lambda} g(\boldsymbol{x})^{\lambda}.$$

Hölder's inequality states that

$$\int_{\mathbb{R}^d} k(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \leq \left( \int_{\mathbb{R}^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{1-\lambda} \left( \int_{\mathbb{R}^d} g(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\lambda}.$$

The Prékopa–Leindler inequality requires a special condition, and its sense is the opposite of Hölder's inequality. Observe that, when $\boldsymbol{x} = \boldsymbol{y}$, the condition (12.5.1) on $h$ just requires that $h(\boldsymbol{x}) \geq k(\boldsymbol{x})$. But $h$ must also satisfy the condition (12.5.1) for all pairs $(\boldsymbol{x}, \boldsymbol{y})$, so it will typically be somewhat larger than $k$.

Remark 12.5.2 (History). The Prékopa–Leindler inequality is part of a larger family of inequalities that dates back to Liusternik's work. This family is now known as the Borell–Brascamp–Lieb inequalities. These inequalities replace the geometric means in Theorem 12.5.1 with other power means.

Liusternik's result involved a power mean with a specific positive exponent. His argument contained some errors, arising from measure-theoretic issues. In 1952, Henstock & MacBeath corrected the proof and developed a larger family of similar inequalities (for all $p > 0$). Later, around 1970, Prékopa and Leindler independently established Theorem 12.5.1, which is the $p = 0$ case. Soon after, Borell and Brascamp–Lieb independently obtained the result for the remaining exponents ($p < 0$).

### 12.5.2 Proof of Brunn–Minkowski from Prékopa–Leindler

Before we continue with the proof of Theorem 12.5.1, let us explain why it implies the Brunn–Minkowski inequality, Theorem 12.4.1.

Fix a parameter $\lambda \in (0,1)$. The cases $\lambda \in \{0,1\}$ are trivial. Let $\mathsf{A}$ and $\mathsf{E}$ be nonempty measurable sets in $\mathbb{R}^d$. If either one is empty, the result is trivial.

Consider the 0–1 indicator functions $f = \mathbb{1}_{\mathsf{A}}$ and $g = \mathbb{1}_{\mathsf{E}}$. Construct the indicator $h = \mathbb{1}_{(1-\lambda)\mathsf{A}+\lambda\mathsf{E}}$ of the convex combination. For all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, it is easy to see that

$$\mathbb{1}_{(1-\lambda)\mathsf{A}+\lambda\mathsf{E}}((1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}) \geq \mathbb{1}_{\mathsf{A}}(\boldsymbol{x})^{1-\lambda} \cdot \mathbb{1}_{\mathsf{E}}(\boldsymbol{y})^{\lambda}.$$

Using the Prékopa–Leindler inequality, Theorem 12.5.1, we calculate that

$$\begin{aligned} \mathrm{Vol}_d((1-\lambda)\mathsf{A} + \lambda\mathsf{E}) &= \int_{\mathbb{R}^d} \mathbb{1}_{(1-\lambda)\mathsf{A}+\lambda\mathsf{E}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\ &\geq \left( \int_{\mathbb{R}^d} \mathbb{1}_{\mathsf{A}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{1-\lambda} \left( \int_{\mathbb{R}^d} \mathbb{1}_{\mathsf{E}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\lambda} = \mathrm{Vol}_d(\mathsf{A})^{1-\lambda} \cdot \mathrm{Vol}_d(\mathsf{E})^{\lambda}. \end{aligned}$$

This is the Brunn–Minkowski inequality.

### 12.5.3 Proof of Prékopa-Leindler

Let us establish the Prékopa–Leindler inequality. The proof is based on a simple measure transportation argument. The next section contains some more context for this construction.

The proof is by induction on the dimension. The difficult case is $d = 1$. Assume $f, g : \mathbb{R} \to \mathbb{R}_{++}$ are strictly positive, continuous functions. The general case follows by approximation.

Introduce the positive normalizing constants

$$F = \int_{\mathbb{R}} f(x)\,\mathrm{d}x \quad \text{and} \quad G = \int_{\mathbb{R}} g(x)\,\mathrm{d}x.$$

Construct functions $S, T : (0,1) \to \mathbb{R}$ that satisfy

$$\frac{1}{F}\int_{-\infty}^{S(a)} f(x)\,\mathrm{d}x = a \quad \text{and} \quad \frac{1}{G}\int_{-\infty}^{T(a)} g(x)\,\mathrm{d}x = a \quad \text{for } a \in (0,1).$$

That is, $S(a)$ and $T(a)$ are the $a$th quantiles of the probability distributions $f/F$ and $g/G$. Since $f$ and $g$ are continuous and positive, $u$ and $v$ are differentiable and increasing. By the fundamental theorem of calculus,

$$\frac{1}{F} f(S(a)) \cdot S'(a) = 1 \quad \text{and} \quad \frac{1}{G} g(T(a)) \cdot T'(a) = 1.$$

Next, define the function $R = (1-\lambda)S + \lambda T$. For $a \in (0,1)$, we have

$$\begin{aligned} R'(a) &= (1-\lambda)S'(a) + \lambda T'(a) \\ &\geq (S'(a))^{1-\lambda} \cdot (T'(a))^{\lambda} \\ &= \left(\frac{F}{f(S(a))}\right)^{1-\lambda} \left(\frac{G}{g(T(a))}\right)^{\lambda}. \end{aligned}$$

The inequality is the inequality between geometric means and arithmetic means.

We are now prepared to bound the integral of the function $h$. Under the change of variables $x = R(a)$, we see that

$$\begin{aligned} \int_{\mathbb{R}} h(x)\,\mathrm{d}x &\geq \int_0^1 h(R(a)) \cdot R'(a)\,\mathrm{d}a \\ &\geq \int_0^1 (f(S(a)))^{1-\lambda} g(T(a))^{\lambda} \cdot \left(\frac{F}{f(S(a))}\right)^{1-\lambda} \left(\frac{G}{g(T(a))}\right)^{\lambda} \mathrm{d}a \\ &= F^{1-\lambda} G^{\lambda} \\ &= \left(\int_{\mathbb{R}} f(x)\,\mathrm{d}x\right)^{1-\lambda} \left(\int_{\mathbb{R}} g(x)\,\mathrm{d}x\right)^{\lambda}. \end{aligned}$$

This completes the proof of the Prékopa–Leindler inequality on $\mathbb{R}$.

Next, assume the Prékopa–Leindler inequality holds for dimension $d-1$. Fix $\lambda \in (0,1)$, and suppose that $f, g, h : \mathbb{R}^d \to \mathbb{R}_+$ satisfying the hypothesis (12.5.1). For points $a, b \in \mathbb{R}$, we set $c = (1-\lambda)a + \lambda b$. Define the functions $f_a, g_b, h_c : \mathbb{R}^{d-1} \to \mathbb{R}_+$ by slicing:

$$f_a(\boldsymbol{x}) = f(\boldsymbol{x}, a); \qquad g_b(\boldsymbol{y}) = g(\boldsymbol{y}, b); \qquad h_c(\boldsymbol{z}) = h(\boldsymbol{z}, c) \qquad \text{for } \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^{d-1}.$$

By the hypothesis (12.5.1), these functions are related via the rule

$$h_c((1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}) \geq f_a(\boldsymbol{x})^{1-\lambda} g_b(\boldsymbol{y})^{\lambda}.$$

Next, define univariate functions $F, G, H : \mathbb{R} \to \mathbb{R}_+$:

$$F(a) = \int_{\mathbb{R}^{d-1}} f_a(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}; \qquad G(b) = \int_{\mathbb{R}^{d-1}} g_b(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}; \qquad H(c) = \int_{\mathbb{R}^{d-1}} h_c(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}.$$

Since $f_a$, $g_b$, and $h_c$ satisfy the hypothesis of the Prékopa–Leindler inequality on $\mathbb{R}^{d-1}$, it follows that

$$H(c) = H((1-\lambda)a + \lambda b) \geq F(a)^{1-\lambda} G(b)^{\lambda}.$$

Therefore, the hypothesis (12.5.1) of the one-dimensional Prékopa–Leindler inequality holds for the functions $F$, $G$, and $H$. By Fubini's theorem and an application of the inequality for $d = 1$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\, h(\boldsymbol{x}) &= \int_{\mathbb{R}} \mathrm{d}c \int_{\mathbb{R}^{d-1}} \mathrm{d}\boldsymbol{z}\, h_c(\boldsymbol{z}) \\
&= \int_{\mathbb{R}} \mathrm{d}c\, H(c) \\
&\geq \left( \int_{\mathbb{R}} \mathrm{d}a\, F(a) \right)^{1-\lambda} \left( \int_{\mathbb{R}} \mathrm{d}b\, G(b) \right)^{\lambda} \\
&= \left( \int_{\mathbb{R}} \mathrm{d}a \int_{\mathbb{R}^{d-1}} \mathrm{d}\boldsymbol{x}\, f_a(\boldsymbol{x}) \right)^{1-\lambda} \left( \int_{\mathbb{R}} \mathrm{d}b \int_{\mathbb{R}^{d-1}} \mathrm{d}\boldsymbol{y}\, g_b(\boldsymbol{y}) \right)^{\lambda} \\
&= \left( \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \right)^{1-\lambda} \left( \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\, g(\boldsymbol{x}) \right)^{\lambda}.
\end{aligned}$$

The last equality follows from Fubini's theorem.

### 12.5.4 Optimal Transport

Here is some background on the optimal transport argument that appears in the proof of the Prékopa–Leindler inequality.

Let $\mu, \nu : \mathbb{R} \to \mathbb{R}$ be probability measures with continuous densities $u$ and $v$. We can think about each measure as describing how one ton of sand is spread across the real line. A *transportation map* is a measurable function $T : \mathbb{R} \to \mathbb{R}$ that reshapes the first pile of sand into the second. That is,

$$\mu(T^{-1}(\mathsf{A})) = \nu(\mathsf{A}) \quad \text{for each measurable set } \mathsf{A} \subset \mathbb{R}.$$

Equivalently,

$$\int \varphi(T(x))\,\mathrm{d}\mu(x) = \int \varphi(x)\,\mathrm{d}\nu(x) \quad \text{for all bounded continuous } \varphi : \mathbb{R} \to \mathbb{R}.$$

We are interesting in finding a transportation map that rearranges the first pile $\mu$ of sand into the form $\nu$ as inexpensively as possible. In particular, we want to minimize the cost

$$\int_{\mathbb{R}} |x - T(x)|^2\,\mathrm{d}\mu(x).$$

This expression says that the cost of moving a unit of sand from the location $x$ to the location $T(x)$ is the *square* of the distance that we move the sand. The integral gives the total cost of moving all of the sand. This is called an *optimal transport* problem.

On the real line, this problem is straightforward to solve because it never pays to move sand farther than we have to. Roughly, we start with the unit of sand that $\mu$ places at the left-most end of the line, and we redistribute it to the left-most point where $\nu$ distributes a unit of sand. We proceed rightward in this fashion. More rigorously, we need to find an increasing function $T : \mathbb{R} \to \mathbb{R}$ where

$$\mathbb{P}_{X\sim\mu}\{X \leq T(a)\} = \int_{-\infty}^{T(a)} v(x)\,\mathrm{d}x = \int_{-\infty}^{a} v(y)\,\mathrm{d}y = \mathbb{P}_{Y\sim\nu}\{Y \leq t\} \quad \text{for all } a \in \mathbb{R}.$$

Differentiating, we see that

$$u(T(a)) \cdot T'(a) = v(a).$$

The optimal transport map can be obtained explicitly by solving this equation.

In the proof of the Prékopa–Leindler inequality, we constructed probability densities by normalizing the two functions $f, g$. It turns out that the worst case is when each of these densities is the uniform distribution on the unit interval $[0, 1]$. To take advantage of this insight, we transported both of the probability densities to the uniform distribution. Although we used the form of the transport map in the argument, we do not directly apply the fact that it minimizes the cost of transport.

The same proof strategy will appear again when we present Barthe's proof of the Brascamp–Lieb inequality. (This is a *different* Brascamp–Lieb inequality than the one we mentioned earlier!)

# Lecture 13: Steiner Symmetrization

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*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*13 November 2018*

## 13.1 Agenda for Lecture 13

In this lecture, we will give a proof of the isoperimetric theorem via Steiner symmetrization, which is a method that decreases the surface area of a convex body while keeping the volume the same. We will define Steiner symmetrization of a convex body, give its properties, and then prove the isoperimetric inequality by way of the sphericity theorem of Gross.

1. Motivation
2. Steiner symmetrization
3. The isoperimetric inequality via symmetrization
4. Proof of the sphericity theorem of Gross

## 13.2 Motivation

Jakob Steiner (1796-1863) was a Swiss mathematician who worked primarily on geometry. Steiner was obsessed with the isoperimetric theorem. He hated using algebra and analysis, and he published five different, purely geometric proofs of the isoperimetric theorem. However, all of his proofs were wrong—or at least incomplete—because he assumed that the isoperimetric problem has a solution.

The mathematician Oskar Perron caricatured Steiner's approach to the isoperimetric theorem as follows.

**Theorem 13.2.1.** *Among all closed curves in the plane with a given length, the circle encloses the maximum area.*

*"Proof."* For any closed curve that is not a circle, there is a method (given by Steiner) that increases the area enclosed by the curve while keeping the length constant. Therefore, the circle has the greatest area among all closed curves. □

To illustrate his complaint, Perron drew an analogy with another flawed argument:

**"Theorem" 13.2.2.** *The number* 1 *is the greatest positive integer.*

*"Proof."* For any positive integer other than 1, there is a method (squaring) that increases the integer. Therefore, 1 is the greatest integer. □

The claim is obviously wrong. The main issue, of course, is that by squaring integers over and over again, we do not obtain a convergent sequence with the limit 1. In contrast, we can correct Steiner's approach to the isoperimetric theorem by demonstrating that his method generates a sequence of figures that converges to the circle.

Despite these flaws, Steiner's geometric methods have enduring interest. Among them, Steiner symmetrization and the four-hinge method are the most famous ones. In today's

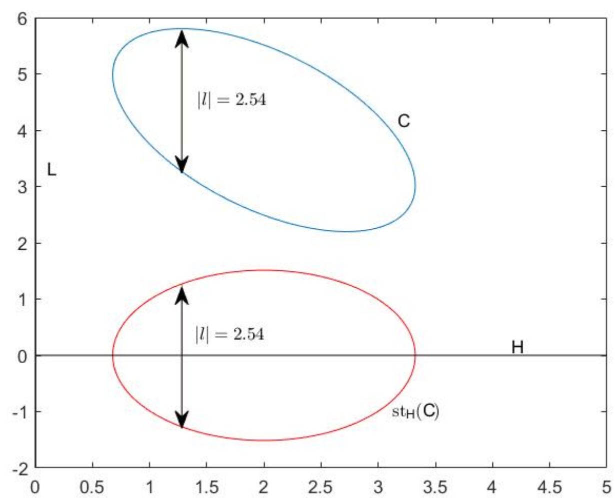


**Figure 13.1** (Steiner symmetrization). The Steiner symmetrization, $\mathrm{st}_{\mathsf{H}}(\mathsf{C})$, of the set $\mathsf{C}$ is symmetric about the hyperplane $\mathsf{H}$.

lecture, we will develop Steiner's symmetrization method and show how it leads to a complete proof of the isoperimetric inequality. The Blaschke selection theorem furnishes the convergence claim, which allows us to avoid Steiner's pitfall.

## 13.3 Steiner Symmetrization

Steiner symmetrization is a method for decreasing the surface area of a set while preserving its volume. In our treatment, we focus on the convex case, although the idea has wider currency.

**Definition 13.3.1** (Steiner symmetrization)**.** Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty convex body, and let $\mathsf{H} \subset \mathbb{R}^d$ be a hyperplane. Let $\mathsf{L}$ be the line normal to $\mathsf{H}$ that passes through the origin. The *Steiner symmetrization* of $\mathsf{C}$ with respect to $\mathsf{H}$ is denoted as $\mathrm{st}_{\mathsf{H}}(\mathsf{C})$. We construct $\mathrm{st}_{\mathsf{H}}(\mathsf{C})$ by taking each nonempty segment $\mathsf{C} \cap (\mathsf{L} + \boldsymbol{x})$ with $\boldsymbol{x} \in \mathsf{H}$ and sliding it down the line $\mathsf{L} + \boldsymbol{x}$ until the midpoint of the segment is located at $\boldsymbol{x} \in \mathsf{H}$. Algebraically,

$$\mathrm{st}_{\mathsf{H}}(\mathsf{C}) := \left\{ \boldsymbol{x} + \boldsymbol{l} \in \mathbb{R}^d : \boldsymbol{x} \in \mathsf{H}, \boldsymbol{l} \in \mathsf{L}, \|\boldsymbol{l}\|_2 \leq \frac{1}{2} \,\mathrm{length}\,(\mathsf{C} \cap (\mathsf{L} + \boldsymbol{x})) \right\}.$$

See Figure 13.1 for an illustration.

### 13.3.1 Properties of the Steiner Symmetrization

Steiner symmetrization has a number of remarkable properties. The following result collects the ones that we will need, but there are others.

**Proposition 13.3.2** (Steiner symmetrization)**.** *Let $\mathsf{H} \subset \mathbb{R}^d$ be an arbitrary hyperplane. For legibility, we write $\mathrm{st}(\cdot)$ in place of $\mathrm{st}_{\mathsf{H}}(\cdot)$. For convex bodies $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$, the operation of Steiner symmetrization...*

1. *Preserves containment.* *If* $\mathsf{C} \subset \mathsf{K}$, *then* $\mathrm{st}(\mathsf{C}) \subset \mathrm{st}(\mathsf{K})$.
2. *Preserves dilations.* *For* $\lambda \geq 0$, *we have* $\mathrm{st}(\lambda\mathsf{C}) = \lambda\,\mathrm{st}(\mathsf{C}) + \boldsymbol{x}_0$ *for some* $\boldsymbol{x}_0 \in \mathbb{R}^d$.
3. *Preserves convexity.* *The symmetrization* $\mathrm{st}(\mathsf{C})$ *is a convex body.*
4. *Reduces Minkowski sums.* *The symmetrization* $\mathrm{st}(\mathsf{C}+\mathsf{K}) \supset \mathrm{st}(\mathsf{C}) + \mathrm{st}(\mathsf{K}) + \boldsymbol{x}_0$ *for some* $\boldsymbol{x}_0 \in \mathbb{R}^d$.
5. *Preserves volume.* *The volume* $\mathrm{Vol}_d(\mathrm{st}(\mathsf{C})) = \mathrm{Vol}_d(\mathsf{C})$.
6. *Decreases surface area.* *The surface area* $S_{d-1}(\mathrm{st}(\mathsf{C})) \leq S_{d-1}(\mathsf{C})$.
7. *Is continuous on proper bodies.* $\mathrm{st}(\cdot)$ *is Hausdorff continuous on the class of convex bodies with positive volume.*

*Proof.* Items 1 and 2 are left as an easy exercise. Let us establish the remaining results.

3. Convexity. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{st}(\mathsf{C})$, and let $\mathsf{L}$ be the line normal to $\mathsf{H}$ that passes through the origin. Consider the trapezoid

$$\mathsf{T} = \mathrm{conv}\{(\mathsf{C} \cap (\mathsf{L}+\boldsymbol{x})) \cup (\mathsf{C} \cap (\mathsf{L}+\boldsymbol{y}))\}.$$

(Draw a picture!) Then the symmetrization $\mathrm{st}(\mathsf{T})$ is also a convex trapezoid, and $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{st}(\mathsf{T})$. Thus, the line segment $[\boldsymbol{x}, \boldsymbol{y}]$ with endpoints $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfies

$$[\boldsymbol{x}, \boldsymbol{y}] \subset \mathrm{st}(\mathsf{T}) \subset \mathrm{st}(\mathsf{C}),$$

where the first inclusion follows from the convexity of $\mathrm{st}(\mathsf{T})$, and the second inclusion follows from Property 1. The proof of compactness is left as an exercise.

4. Minkowski sums. Without loss of generality, assume that $\mathsf{H}$ contains the origin, and let $\boldsymbol{x} \in \mathrm{st}(\mathsf{C})$ and $\boldsymbol{y} \in \mathrm{st}(\mathsf{K})$. We write

$$\boldsymbol{x} = \boldsymbol{g} + \boldsymbol{l} \quad \text{and} \quad \boldsymbol{y} = \boldsymbol{h} + \boldsymbol{m} \quad \text{where } \boldsymbol{g}, \boldsymbol{h} \in \mathsf{H} \text{ and } \boldsymbol{l}, \boldsymbol{m} \in \mathsf{L}.$$

By definition of Steiner symmetrization,

$$\|\boldsymbol{l}\|_2 \leq \frac{1}{2}\,\mathrm{length}(\mathsf{C} \cap (\mathsf{L}+\boldsymbol{x})) \quad \text{and} \quad \|\boldsymbol{m}\|_2 \leq \frac{1}{2}\,\mathrm{length}(\mathsf{K} \cap (\mathsf{L}+\boldsymbol{y})).$$

We can bound the total length of the normal vectors by a short calculation:

$$\begin{aligned}
\|\boldsymbol{l}+\boldsymbol{m}\|_2 &\leq \|\boldsymbol{l}\|_2 + \|\boldsymbol{m}\|_2 && (13.3.1)\\
&\leq \frac{1}{2}\,\mathrm{length}(\mathsf{C} \cap (\mathsf{L}+\boldsymbol{x})) + \frac{1}{2}\,\mathrm{length}(\mathsf{K} \cap (\mathsf{L}+\boldsymbol{y}))\\
&= \frac{1}{2}\,\mathrm{length}(\mathsf{C} \cap (\mathsf{L}+\boldsymbol{x}) + \mathsf{K} \cap (\mathsf{L}+\boldsymbol{y})) && (13.3.2)\\
&= \frac{1}{2}\,\mathrm{length}((\mathsf{C}-\boldsymbol{x}) \cap \mathsf{L} + (\mathsf{K}-\boldsymbol{y}) \cap \mathsf{L})\\
&\leq \frac{1}{2}\,\mathrm{length}((\mathsf{C}+\mathsf{K}-\boldsymbol{x}-\boldsymbol{y}) \cap \mathsf{L}) && (13.3.3)\\
&= \frac{1}{2}\,\mathrm{length}((\mathsf{C}+\mathsf{K}) \cap (\mathsf{L}+\boldsymbol{x}+\boldsymbol{y})).
\end{aligned}$$

The first line (13.3.1) is just the triangle inequality. Step (13.3.2) follows from the fact that the the length of the sum of two parallel line segments is the sum of the lengths of the line segments. Last, (13.3.3) holds because $(\mathsf{A} \cap \mathsf{C}) + (\mathsf{B} \cap \mathsf{C}) \subset (\mathsf{A} + \mathsf{B}) \cap \mathsf{C}$. The remaining steps follow from elementary geometric reasoning.

We obviously have

$$\boldsymbol{x} + \boldsymbol{y} = (\boldsymbol{g} + \boldsymbol{h}) + (\boldsymbol{l} + \boldsymbol{m}) \quad \text{where } \boldsymbol{g} + \boldsymbol{h} \in \mathsf{C} + \mathsf{K} \text{ and } \boldsymbol{l} + \boldsymbol{m} \in \mathsf{L}.$$

The preceding argument implies that $\boldsymbol{x} + \boldsymbol{y} \in \mathrm{st}(\mathsf{C} + \mathsf{K})$.

5. Volume. This property follows instantly from the Cavalieri's principle, which states that two bodies have the same volume if their sections have the same volume. Equivalently, we can compute the volume by writing down the integral and applying Fubini's theorem.

6. Surface area. By Property 4, we have

$$\mathrm{st}(\mathsf{C} + \varepsilon \mathsf{B}_d) \supset \mathrm{st}(\mathsf{C}) + \mathrm{st}(\mathsf{B}_d) = \mathrm{st}(\mathsf{C}) + \mathsf{B}_d.$$

The Minkowski surface area formula, given in Lecture 12 (Definition 3.1), yields

$$\begin{aligned} S_{d-1}(\mathrm{st}(\mathsf{C}) &= \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\mathrm{st}(\mathsf{C}) + \varepsilon \mathsf{B}_d) - \mathrm{Vol}_d(\mathrm{st}(\mathsf{C}))}{\varepsilon} \\ &\leq \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\mathrm{st}(\mathsf{C} + \varepsilon \mathsf{B}_d)) - \mathrm{Vol}_d(\mathrm{st}(\mathsf{C}))}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\mathsf{C} + \varepsilon \mathsf{B}_d) - \mathrm{Vol}_d(\mathsf{C})}{\varepsilon} = S_{d-1}(\mathsf{C}). \end{aligned}$$

The inequality is an immediate consequence of the inclusion in the preceding display.

7. Continuity. Let $\mathsf{C}_j$ be convex bodies with *positive* volume such that $\mathsf{C}_j \to \mathsf{C}$. Without loss of generality, assume that $\mathbf{0} \in \mathrm{int}(\mathsf{C})$ and that $\mathsf{H}$ passes through $\mathbf{0}$. Under these assumptions, Hausdorff convergence is equivalent to the condition that, for every $\varepsilon > 0$ and large enough $j$,

$$(1 - \varepsilon)\mathsf{C} \subset \mathsf{C}_j \subset (1 + \varepsilon)\mathsf{C}.$$

By Property 1, it follows that

$$(1 - \varepsilon)\,\mathrm{st}(\mathsf{C}) \subset \mathrm{st}(\mathsf{C}_j) \subset (1 + \varepsilon)\,\mathrm{st}(\mathsf{C}).$$

Since $\mathbf{0} \in \mathrm{st}(\mathsf{C})$, this condition is equivalent to the limit $\mathrm{st}(\mathsf{C}_j) \to \mathrm{st}(\mathsf{C})$.

This completes the argument. □

## 13.4 The Isoperimetric Inequality via Symmetrization

It is intuitive that repeated symmetrization of a convex body can generate a sequence that converges to a Euclidean ball. This result is called the sphericity theorem, and it was established by Gross in 1917. We will state the result formally, and then show how it implies the isoperimetric inequality.

**Theorem 13.4.1** (Gross sphericity). *Let* $\mathsf{C} \in \mathbb{R}^d$ *be a convex body with positive volume and let* $\mathrm{Steiner}(\mathsf{C})$ *denote the set of all the convex bodies obtained from* $\mathsf{C}$ *by a finite number of successive Steiner symmetrizations through hyperplanes containing the origin. Then* $\mathrm{Steiner}(\mathsf{C})$ *contains a sequence that converges to a scaled Euclidean ball.*

The proof of Theorem 13.4.1 appears below in Section 13.5.

To establish the isoperimetric inequality, we rely on an easy corollary of the sphericity theorem. This result also depends on the fact that symmetrization preserves volume.

**Corollary 13.4.2.** *There is a sequence of hyperplanes* $\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3, \dots$ *through the origin for which*

$$\mathrm{st}_{\mathsf{H}_j}\,\mathrm{st}_{\mathsf{H}_{j-1}}\dots\mathrm{st}_{\mathsf{H}_1}(\mathsf{C}) \to \left(\frac{\mathrm{Vol}_d(\mathsf{C})}{\mathrm{Vol}_d(\mathsf{B}_d)}\right)^{1/d}\mathsf{B}_d.$$

With Corollary 13.4.1 at hand, we can prove a version of the isoperimetric inequality for convex bodies.

**Corollary 13.4.3** (Isoperimetric inequality). *For a convex body* $\mathsf{C} \subset \mathbb{R}^d$ *with positive volume,*

$$\left(\frac{S_{d-1}(\mathsf{C})}{S_{d-1}(\mathsf{B}_d)}\right)^{1/(d-1)} \geq \left(\frac{\mathrm{Vol}_d(\mathsf{C})}{\mathrm{Vol}_d(\mathsf{B}_d)}\right)^{1/d}.$$

*Proof.* Let $\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3, \dots$ be the sequence of hyperplanes guaranteed by Corollary 13.4.2. By Property 6 of Steiner symmetrization,

$$S_{d-1}(\mathsf{C}) \geq S_{d-1}(\mathrm{st}_{\mathsf{H}_1}(\mathsf{C})) \geq \cdots \geq S_{d-1}(\mathrm{st}_{\mathsf{H}_j}\dots\mathrm{st}_{\mathsf{H}_1}(\mathsf{C})) \geq \cdots.$$

Passing to the limit,

$$S_{d-1}(\mathsf{C}) \geq S_{d-1}\left(\left(\frac{\mathrm{Vol}_d(\mathsf{C})}{\mathrm{Vol}_d(\mathsf{B}_d)}\right)^{1/d}\mathsf{B}_d\right) = \left(\frac{\mathrm{Vol}_d(\mathsf{C})}{\mathrm{Vol}_d(\mathsf{B}_d)}\right)^{(d-1)/d} S_{d-1}(\mathsf{B}_d).$$

We have used the continuity of the Minkowski surface area $S_{d-1}$. □

## 13.5 Proof of the Sphericity Theorem of Gross

We continue with the proof of Theorem 13.4.1. For a convex body $\mathsf{K} \subset \mathbb{R}^d$, we define the radius of the smallest origin-centered ball that contains the body:

$$\varrho(\mathsf{K}) := \inf\{\lambda \geq 0 : \mathsf{K} \subset \lambda\mathsf{B}_d\}.$$

Let $\varrho_\infty$ be the minimum radius of a ball that contains an element of $\mathrm{Steiner}(\mathsf{C})$:

$$\varrho_\infty = \inf\{\varrho(\mathsf{K}) : \mathsf{K} \in \mathrm{Steiner}(\mathsf{C})\}.$$

By Property 1 of Steiner symmetrization, the set $\mathrm{Steiner}(\mathsf{C})$ is bounded in Hausdorff metric. Therefore, by Blaschke's selection theorem, there exists a convergent sequence $\{\mathsf{K}_j : j \in \mathbb{N}\} \subset \mathrm{Steiner}(\mathsf{C})$ with the properties that

$$\mathsf{K}_j \to \mathsf{K}_\infty \quad \text{and} \quad \varrho_\infty = \lim_{j\to\infty} \varrho(\mathsf{K}_j) = \varrho(\mathsf{K}_\infty).$$

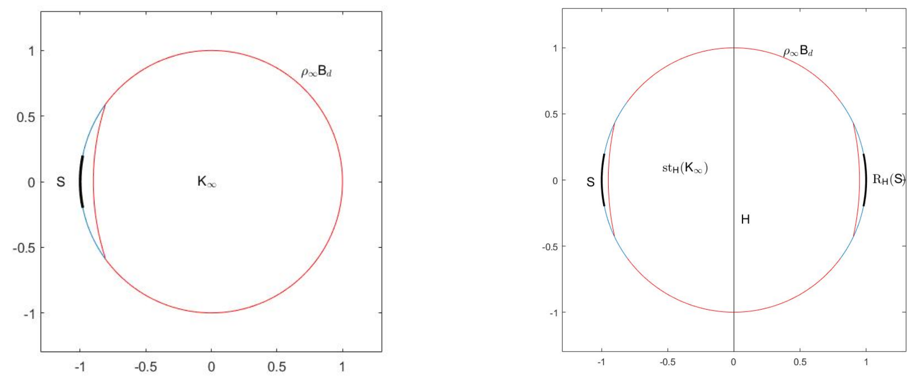


**Figure 13.2** (Proof of the sphericity theorem). Assume that there exists a spherical cap $\mathsf{S}$ that does not intersect $\mathsf{K}_\infty$. When we apply Steiner symmetrization, the reflection $\boldsymbol{R}_{\mathsf{H}}(\mathsf{S})$ does not intersect $\mathrm{st}_{\mathsf{H}}(\mathsf{C})$.

We have used the continuity of the radius $\varrho$ to obtain the last identity.

The set $\mathsf{K}_\infty$ is contained in the ball $\varrho_\infty \mathsf{B}_d$. For the sake of contradiction, suppose that $\mathsf{K}_\infty$ is a proper subset of $\varrho_\infty \mathsf{B}_d$. Since $\mathsf{K}_\infty$ is a convex body, there exists a nondegenerate spherical cap $\mathsf{S} \subset \varrho_\infty \mathbb{S}^{d-1}$ that does not meet the set: $\mathsf{S} \cap \mathsf{K}_\infty = \emptyset$. (Recall that a spherical cap is the intersection of a Euclidean ball with the sphere, and nondegeneracy means that the cap is neither empty nor a single point.) We will argue that it is possible to symmetrize the set $\mathsf{K}_\infty$ repeatedly to ensure that it does not touch the sphere $\varrho_\infty \mathbb{S}^{d-1}$ anywhere.

In the sequel, every hyperplane $\mathsf{H}$ will be understood to contain the origin. Define $\boldsymbol{R}_{\mathsf{H}}(\cdot)$ to be the reflection with respect to a hyperplane $\mathsf{H}$. For every hyperplane $\mathsf{H}$,

$$\mathsf{S} \cap \mathrm{st}_{\mathsf{H}}(\mathsf{K}_\infty) = \emptyset \quad \text{and} \quad \boldsymbol{R}_{\mathsf{H}}(\mathsf{S}) \cap \mathrm{st}_{\mathsf{H}}(\mathsf{K}_\infty) = \emptyset.$$

See Figure 13.2 for an illustration. By compactness of the sphere, we can cover $\varrho_\infty \mathbb{S}^{d-1}$ with the reflections of the spherical cap $\mathsf{S}$ in a finite number of hyperplanes. Therefore, we can find a finite list $\mathsf{H}_1, \dots, \mathsf{H}_m$ of hyperplanes with the property that

$$\varrho_\infty \mathbb{S}^{d-1} \subset \bigcup_{i=1}^m \boldsymbol{R}_{\mathsf{H}_i}(\mathsf{S}).$$

Use these hyperplanes to define the successive symmetrization $\mathrm{st}_\star := \mathrm{st}_{\mathsf{H}_m} \mathrm{st}_{\mathsf{H}_{m-1}} \dots \mathrm{st}_{\mathsf{H}_1}$.

Now, this discussion ensures that

$$\varrho_\infty \mathbb{S}^{d-1} \cap \mathrm{st}_\star(\mathsf{K}_\infty) = \emptyset.$$

Therefore, $\mathrm{st}_\star(\mathsf{K}_\infty) \subset \mathrm{int}(\varrho_\infty \mathsf{B}_d)$. Since the set $\mathrm{st}_\star(\mathsf{K}_\infty)$ is compact,

$$\varrho(\mathrm{st}_\star(\mathsf{K}_\infty)) < \varrho_\infty.$$

For each index $j$, define the proper convex body $\mathsf{C}_j := \mathrm{st}_\star(\mathsf{K}_j) \in \mathrm{Steiner}(\mathsf{C})$. By the definition of $\varrho_\infty$, we have $\varrho(\mathsf{C}_j) \geq \varrho_\infty$ for each index $j$. Continuity of the symmetrization operator implies that

$$\lim_{j\to\infty} \mathsf{C}_j = \lim_{j\to\infty} \mathrm{st}_\star(\mathsf{K}_j) = \mathrm{st}_\star(\mathsf{K}_\infty).$$

Since the radius $\varrho$ is continuous, this identity implies that

$$\varrho_\infty \le \lim_{j\to\infty} \varrho(\mathsf{C}_j) = \varrho(\mathrm{st}_\star(\mathsf{K}_\infty)) < \varrho_\infty.$$

This is a blatant contradiction.

# Lecture 14: The John Ellipsoid

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*ACM 204, Fall 2018*
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*15 November 2018*

## 14.1 Agenda for Lecture 14

This lecture introduces the theory of *ellipsoids*, which are the affine images of the Euclidean unit ball. We consider the problem of approximating a convex body by the maximum-volume ellipsoid that can be inscribed inside the body. Every convex body admits an affine transformation where the maximum-volume ellipsoid is the Euclidean ball. We will develop a characterization of convex bodies that are subject to this normalization. Finally, we give a simple application of this result to Banach space geometry. In the next lecture, we will use these tools (and more) to obtain a *reversed* form of the isoperimetric inequality.

Here is a summary of the topics:

1. Affine classes.
2. Ellipsoids
3. The John ellipsoid
4. Characterization of John's position
5. Equivalence of norms

## 14.2 The Affine Class of a Convex Body

Convex bodies are a diverse club. Some are pointy, some have lots of facets, some are very thin, some are very thick. See Figure 14.1 for examples.

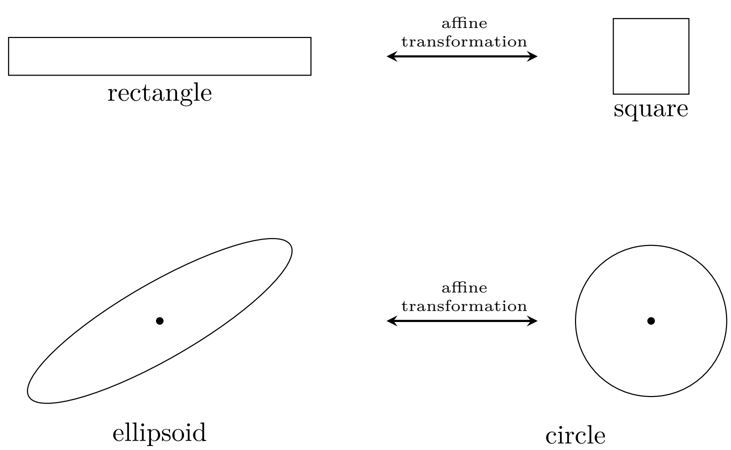


**Figure 14.1** (Affine transformations of convex bodies). Some examples of convex bodies and their affine images.

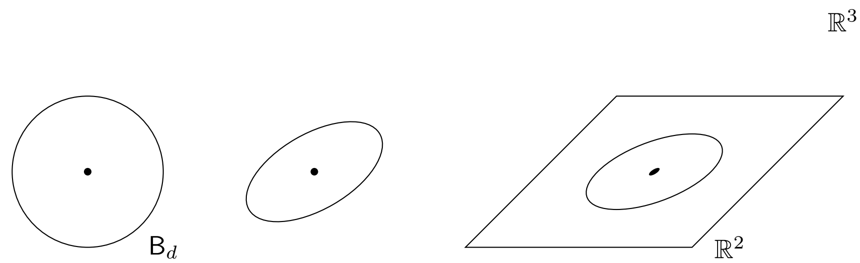


**Figure 14.2** (Examples of ellipsoids). The first two ellipsoids are nondegenerate Euclidean balls in $\mathbb{R}^2$. The third is a degenerate ellipsoid in $\mathbb{R}^3$.

We can go from the figures on the left to the figures on the right (and *vice versa*) by means of an affine transformation. The figures on the right have a larger volume relative to their surface area than the ones on the left. We will be interested in affine transformations, like these, that improve the isoperimetric ratio.

A nondegenerate affine transformation preserves some basic geometric features of a convex body, including convexity and boundary structure. Therefore, it is sometimes appropriate to group all of the (nondegenerate) affine images of a convex body into an equivalence class.

## 14.3 Ellipsoids

We begin our treatment of affine families of convex bodies with the most basic example: ellipsoids. As usual, let $\mathsf{B}_d \subset \mathbb{R}^d$ be the Euclidean unit ball, centered at the origin.

**Definition 14.3.1** (Ellipsoid)**.** An *ellipsoid* $\mathsf{E}$ is an affine image of the Euclidean ball $\mathsf{B}_d$:

$$\mathsf{E} = \boldsymbol{T}\mathsf{B}_d + \boldsymbol{a}$$

where $\boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map and the point $\boldsymbol{a} \in \mathbb{R}^d$ is called the *center* of the ellipsoid. We say that the ellipsoid is *nondegenerate* when the linear map $\boldsymbol{T}$ is nonsingular, in which case $\dim \mathsf{E} = d$. See Figure 14.2 for an illustration.

Fix a nondegenerate ellipsoid $\mathsf{E} = \boldsymbol{T}\mathsf{B}_d + \boldsymbol{a}$. We can write

$$\mathsf{E} = \left\{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{T}^{-1}(\boldsymbol{x}-\boldsymbol{a}),\, \boldsymbol{T}^{-1}(\boldsymbol{x}-\boldsymbol{a})\rangle \le 1\right\}$$

This expression can be rewritten again in the form

$$\mathsf{E} = \left\{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{P}(\boldsymbol{x}-\boldsymbol{a}),\, \boldsymbol{x}-\boldsymbol{a}\rangle \le 1\right\}$$

where the matrix $\boldsymbol{P} := (\boldsymbol{T}\boldsymbol{T}^*)^{-1}$ is positive definite.

Since $\boldsymbol{T}$ is nonsingular, we can introduce the eigenvalue factorization $\boldsymbol{T}\boldsymbol{T}^* = \boldsymbol{Q}\boldsymbol{\Lambda}^2\boldsymbol{Q}^*$. The matrix $\boldsymbol{Q}$ is orthogonal with columns $\boldsymbol{q}_1, \dots, \boldsymbol{q}_d$. The matrix $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ is positive definite. Changing variables, we have

$$\begin{aligned}\mathsf{E} &= \left\{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{Q}\boldsymbol{\Lambda}^{-2}\boldsymbol{Q}^*(\boldsymbol{x}-\boldsymbol{a}),\, \boldsymbol{x}-\boldsymbol{a}\rangle \le 1\right\} \\ &= \left\{\boldsymbol{x} \in \mathbb{R}^d : \sum\nolimits_{i=1}^d \lambda_i^{-2}\langle \boldsymbol{q}_i,\, \boldsymbol{x}-\boldsymbol{a}\rangle^2 \le 1\right\},\end{aligned}$$

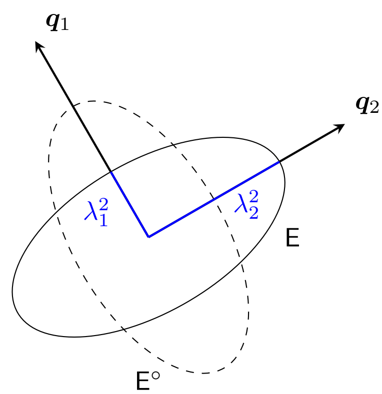


**Figure 14.3** (Ellipsoids and eigenvalue factorizations). The ellipsoid $\mathsf{E}$ has semiaxes in the directions of unit eigenvectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ with lengths corresponding to $\lambda_1^2$ and $\lambda_2^2$, respectively. The semiaxes of the polar ellipsoid $\mathsf{E}^\circ$ have the same directions $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ and the reciprocal lengths $\lambda_1^{-2}$ and $\lambda_2^{-2}$.

We can interpret the eigenvectors $\boldsymbol{q}_i$ and the squares of the eigenvalues $\lambda_i^2$ of the matrix $\boldsymbol{TT}^*$ as the directions and lengths of the semiaxes of $\mathsf{E}$. See Figure 14.3.

We can easily compute the volume of the ellipsoid in terms of the eigenvalues:

$$\begin{aligned} \mathrm{Vol}_d(\mathsf{E}) &= \mathrm{Vol}_d(\boldsymbol{T}\mathsf{B}_d) = |\det \boldsymbol{T}| \cdot \mathrm{Vol}_d(\mathsf{B}_d) \\ &= \left( \prod\nolimits_{i=1}^d \lambda_i \right) \mathrm{Vol}_d(\mathsf{B}_d) = \left( \prod\nolimits_{i=1}^d \lambda_i^2 \right)^{1/2} \mathrm{Vol}_d(\mathsf{B}_d). \end{aligned} \tag{14.3.1}$$

The first identity holds because volume is translation invariant, and the second follows the change of variables formula. We have taken the last step for our later convenience.

The representation of a nondegenerate ellipsoid $\mathsf{E} = \boldsymbol{T}\mathsf{B}_d + \boldsymbol{a}$ in terms of an affine map is not unique. To remove the extra degrees of freedom, consider the polar decomposition $\boldsymbol{T} = \boldsymbol{SU}$ where $\boldsymbol{S}$ is positive definite and $\boldsymbol{U}$ is orthogonal. Then

$$\mathsf{E} = (\boldsymbol{SU})\mathsf{B}_d + \boldsymbol{a} = \boldsymbol{S}(\boldsymbol{U}\mathsf{B}_d) + \boldsymbol{a} = \boldsymbol{S}\mathsf{B}_d + \boldsymbol{a}.$$

The latter representation of the ellipsoid with a positive-definition matrix $\boldsymbol{S}$ is unique.

In the sequel, we will focus on origin-symmetric ellipsoids, which take the form

$$\mathsf{E} = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{Px},\, \boldsymbol{x} \rangle \leq 1 \right\} \quad \text{where } \boldsymbol{P} \text{ is positive definite.}$$

In this case, we can easily obtain a formula for the polar ellipsoid:

$$\mathsf{E}^\circ = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{P}^{-1}\boldsymbol{x},\, \boldsymbol{x} \rangle \leq 1 \right\} = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \sum\nolimits_{i=1}^d \lambda_i^2 \langle \boldsymbol{q}_i,\, \boldsymbol{x} \rangle^2 \leq 1 \right\}. \tag{14.3.2}$$

As before, $\boldsymbol{P}^{-1} = \boldsymbol{Q\Lambda}^2\boldsymbol{Q}^*$. The proof is an exercise.

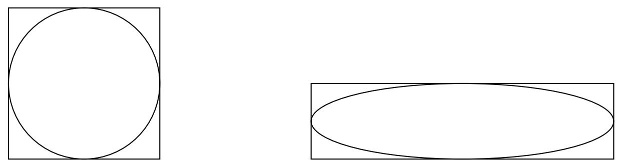

**Figure 14.4** (Maximum volume ellipsoid). The maximum volume ellipsoid contained in each of two simple convex bodies.

## 14.4 The John Ellipsoid

In this section, we study an extremal problem involving ellipsoids and convex bodies: What is the "largest" ellipsoid contained inside a nondegenerate convex body $\mathsf{C}$ in $\mathbb{R}^d$? We will quantify the size of the ellipsoid in terms of its volume. This leads to the formulation

$$\max\{\mathrm{Vol}_d(\mathsf{E}) : \mathsf{E} \subset \mathsf{C} \text{ and } \mathsf{E} \text{ an ellipsoid}\}.$$

We will solve a simplified version of this problem and characterize convex bodies for which the solution is the Euclidean ball. See Figure 14.4 for examples of convex bodies along with their associated maximum volume ellipsoids.

From now on, we will restrict our attention to a norm ball $\mathsf{K} \subset \mathbb{R}^d$. That is, $\mathsf{K}$ is

- origin-symmetric ($\mathsf{K} = -\mathsf{K}$),
- a convex body, and
- contains $\mathbf{0} \in \operatorname{int} \mathsf{K}$. Equivalently, $\dim \mathsf{K} = d$.

For a norm ball, the maximum-volume ellipsoid must be an origin-centered ellipsoid because it inherits the symmetries of $\mathsf{K}$. This is left as an exercise for the reader.

We now prove a basic existence and uniqueness result about maximum-volume ellipsoids.

**Theorem 14.4.1** (Löwner–John). *Let $\mathsf{K} \subset \mathbb{R}^d$ be a norm ball. Among all ellipsoids $\mathsf{E} \subset \mathsf{K}$, there is a unique ellipsoid of maximum volume.*

For the proof, we need an important inequality from matrix theory.

**Fact 14.4.2** (Minkowski determinant theorem). *The function $\boldsymbol{A} \mapsto \log\det \mathsf{A}$ is strictly concave on positive-definite matrices, modulo lines. More precisely, for positive-definite matrices $\boldsymbol{P}, \boldsymbol{S}$, we have the inequality*

$$\log\det\big(\tfrac{1}{2}(\boldsymbol{P}+\boldsymbol{S})\big) \geq \tfrac{1}{2}(\log\det\boldsymbol{P} + \log\det\boldsymbol{S}).$$

*Equality holds if and only if $\boldsymbol{P} = \lambda\boldsymbol{S}$ for some $\lambda > 0$.*

*Proof sketch.* By factorization, we can reduce to the case where $\boldsymbol{P} = \mathbf{I}$:

$$\det\big(\tfrac{1}{2}(\boldsymbol{P}+\boldsymbol{S})\big) = \det(\boldsymbol{P}) \cdot \det\Big(\tfrac{1}{2}(\mathbf{I} + \boldsymbol{P}^{-1/2}\boldsymbol{S}\boldsymbol{P}^{-1/2})\Big) =: \det(\boldsymbol{P}) \cdot \det\big(\tfrac{1}{2}(\mathbf{I}+\boldsymbol{A})\big).$$

We bound the latter determinant as follows.

$$\det\big(\tfrac{1}{2}(\mathbf{I}+\boldsymbol{A})\big) = \prod\nolimits_{i=1}^{d} \tfrac{1}{2}(1+a_i) \geq \Big(1 + \prod\nolimits_{i=1}^{d} a_i\Big)^{1/2} = (\det(\mathbf{I}) + \det(\boldsymbol{A}))^{1/2}.$$

The inequality is an elementary numerical bound. Combine the displays and rewrite in terms of $\boldsymbol{S}$ and $\boldsymbol{P}$. □

With this inequality at hand, we can establish the Löwner–John theorem.

*Proof of the Löwner–John Theorem.* First, we establish the existence of a maximizing ellipsoid. Recall that we can parameterize ellipsoids uniquely as $\mathsf{E} = \boldsymbol{S}\mathsf{B}_d$ where $\boldsymbol{S}$ is a positive-semidefinite matrix. Note that we are allowing degenerate cases. Introduce the set of ellipsoids included in the norm ball $\mathsf{K}$:

$$\mathcal{E} := \{\boldsymbol{S} \in \mathbb{R}^{d\times d} \text{ is psd} : \boldsymbol{S}\mathsf{B}_d \subset \mathsf{K}\}.$$

Since $\mathsf{K}$ is compact, this set of matrices is also compact. Next, observe that the map $\boldsymbol{S} \mapsto \mathrm{Vol}_d(\boldsymbol{S}\mathsf{B}_d)$ is continuous. Therefore, it achieves its maximum on $\mathcal{E}$, say at $\boldsymbol{S}_\star$. Finally, note that the optimizer $\boldsymbol{S}_\star$ is actually positive definite. Indeed, if $\boldsymbol{S}_\star$ is singular, then $\mathrm{Vol}_d(\boldsymbol{S}_\star\mathsf{B}_d) = 0$. This cannot happen because $\mathbf{0} \in \operatorname{int}\mathsf{K}$, which implies that $\mathsf{K}$ contains a ball with positive volume.

Next, we show that the maximizing ellipsoid is unique. Suppose that $\mathsf{E}_1 = \boldsymbol{S}_1\mathsf{B}_d$ and $\mathsf{E}_2 = \boldsymbol{S}_2\mathsf{B}_d$ are different maximum-volume ellipsoids that are contained in $\mathsf{K}$. The ellipsoids have the same volume, so $\det \boldsymbol{S}_1 = \det \boldsymbol{S}_2$. As a consequence, $\boldsymbol{S}_1 \neq \lambda \boldsymbol{S}_2$ for any scalar $\lambda$.

Now, consider the average of the two ellipsoids:

$$\mathsf{E}' := \tfrac{1}{2}(\boldsymbol{S}_1 + \boldsymbol{S}_2)\mathsf{B}_d = \tfrac{1}{2}(\mathsf{E}_1 + \mathsf{E}_2) \subset \mathsf{K}.$$

In other words, $\mathsf{E}'$ is an ellipsoid that is contained in the convex set $\mathsf{K}$. Let us estimate the volume of the ellipsoid $\mathsf{E}'$. In light of the volume computation (14.3.1),

$$\begin{aligned}\mathrm{Vol}_d(\mathsf{E}') &= \left|\det(\tfrac{1}{2}\boldsymbol{S}_1 + \tfrac{1}{2}\boldsymbol{S}_2)\right| \cdot \mathrm{Vol}_d(\mathsf{B}_d) \\ &> (\det \boldsymbol{S}_1)^{1/2}(\det \boldsymbol{S}_2)^{1/2} \cdot \mathrm{Vol}_d(\mathsf{B}_d) \\ &= \max\{\mathrm{Vol}_d(\mathsf{E}_1), \mathrm{Vol}_d(\mathsf{E}_2)\}.\end{aligned}$$

Fact 14.4.2 produces a strict inequality because $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are not on a line. The last relation holds because $\det \boldsymbol{S}_1 = \det \boldsymbol{S}_2$. We arrive at a contradiction. □

We have shown that every norm ball contains a unique maximum-volume ellipsoid, which is known as the *John ellipsoid*. By a similar argument, every norm ball is contained in a unique minimum-volume ellipsoid, which is known as the *Löwner ellipsoid*. We focus on John ellipsoids in our discussion.

An important property of John ellipsoids is that they are affine covariant. If $\mathsf{E}$ is the maximum-volume ellipsoid of $\mathsf{K}$ and $\boldsymbol{S}$ is a positive-definite matrix, then $\boldsymbol{S}\mathsf{E}$ is the maximum-volume ellipsoid of $\boldsymbol{S}\mathsf{K}$. This is an easy exercise.

Suppose that $\mathsf{K}$ is a norm ball with John ellipsoid $\mathsf{E} = \boldsymbol{S}\mathsf{B}_d$, where $\boldsymbol{S}$ is positive definite. It follows that the affine image $\tilde{\mathsf{K}} = \boldsymbol{S}^{-1}\mathsf{K}$ is a norm ball whose John ellipsoid is the Euclidean ball $\mathsf{B}_d$. Thus, every norm ball has a (unique) affine image whose John ellipsoid is the Euclidean ball. See Figure 14.5. This observation motivates a definition.

**Definition 14.4.3** (John's position)**.** A norm ball $\mathsf{K}$ is in *John's position* if its maximum-volume ellipsoid is the Euclidean ball.

To summarize, we have shown that every norm ball has a (unique) affine image in John's position.

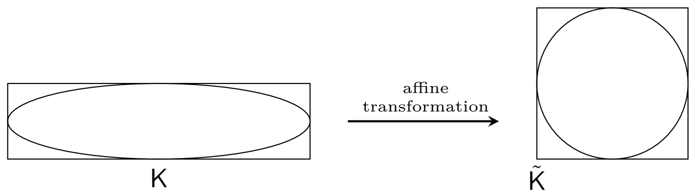


**Figure 14.5** (John's position). The affine transformation taking a convex body K and its maximum-volume ellipsoid to John's position. In John's position, the maximum volume ellipsoid is just the Euclidean ball $\mathsf{B}_d$.

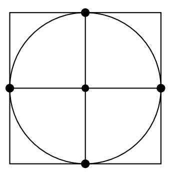

**Figure 14.6** (Contact points). A convex body in John's position along with four contact points.

## 14.5 Characterization of John's Position

Out next goal is to obtain a characterization of norm balls in John's position.

**Theorem 14.5.1** (John's Characterization). *Let $\mathsf{K} \subset \mathbb{R}^d$ be a norm ball with $\mathsf{B}_d \subset \mathsf{K}$. The following are equivalent:*

1. *$\mathsf{K}$ has $\mathsf{B}_d$ as its maximum-volume ellipsoid. That is, $\mathsf{K}$ is in John's position.*
2. *There are unit vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m \in \mathsf{B}_d \cap \operatorname{bd} \mathsf{K}$ and positive weights $\alpha_1, \dots, \alpha_m > 0$ such that*
$$\sum\nolimits_{i=1}^m \alpha_i \, \boldsymbol{u}_i \boldsymbol{u}_i^* = \mathbf{I}_d \quad \textit{and} \quad \sum\nolimits_{i=1}^m \alpha_i = d. \tag{14.5.1}$$

*The number $m$ of contact points can be chosen with $d \leq m \leq \binom{d+1}{2} + 1$.*

The unit vectors $\boldsymbol{u}_i$ in the theorem statement are called *contact points*. They are chosen from among the locations where the inscribed ball $\mathsf{B}_d$ touches the boundary of $\mathsf{K}$, as illustrated in Figure 14.6. The first equation in (14.5.1) requires that the contact points be spread out over the sphere, and they cannot cluster too close to a subspace. The second equation in (14.5.1) is just the trace of the first condition.

It also is clear that, for a norm ball $\mathsf{K}$, can require that the antipode of a point $-\boldsymbol{u}_i$ be included along with the point $\boldsymbol{u}_i$. This observation will play a role in the next lecture.

We begin the argument with a simple geometric lemma.

**Lemma 14.5.2** (Supporting hyperplane at a contact point). *Instate the notation of Theorem 14.5.1. For each index $i$, the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle = 1\}$ supports the norm ball $\mathsf{K}$ at the point $\boldsymbol{u}_i$.*

*Proof.* Since $\boldsymbol{u}_i \in \operatorname{bd} \mathsf{K}$, there is a supporting hyperplane $\mathsf{H} = \{\boldsymbol{x} : \langle \boldsymbol{v}, \boldsymbol{x} \rangle \leq 1\}$ to $\mathsf{K}$ at the point $\boldsymbol{u}_i$. It must be the case that $\boldsymbol{v}$ has unit norm because $\boldsymbol{u}_i$ has unit norm. Since

$\mathsf{B}_d \subset \mathsf{K}$, the hyperplane $\mathsf{H}$ also supports $\mathsf{B}_d$ at $\boldsymbol{u}_i$. But the only supporting hyperplane to the Euclidean ball at the point $\boldsymbol{u}_i$ has the unit normal $\boldsymbol{u}_i$. Thus $\boldsymbol{v} = \boldsymbol{u}_i$. □

Let us continue with the proof of the characterization theorem.

*Proof of John's Characterization.* First, we establish that the second condition implies the first condition. Construct an inscribed ellipsoid

$$\mathsf{E} = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \sum\nolimits_{i=1}^{d} \lambda_i^{-2} \langle \boldsymbol{q}_i,\, \boldsymbol{x} \rangle^2 \leq 1 \right\} \subset \mathsf{K}.$$

We need to show that $\prod_{i=1}^{d} \lambda_i \leq 1$ with equality if and only if $\lambda_i = 1$ for $i = 1, \dots, d$. In view of (14.3.1) and the general discussion of ellipsoids, this condition ensures that $\mathrm{Vol}_d(\mathsf{E}) = \mathrm{Vol}_d(\mathsf{B}_d)$ with equality if and only if $\mathsf{E} = \mathsf{B}_d$.

According to Lemma 14.5.2, the hyperplane $\{\boldsymbol{x} : \langle \boldsymbol{u}_j,\, \boldsymbol{x} \rangle = 1\}$ supports the set $\mathsf{K}$ at the point $\boldsymbol{u}_j$. Since $\mathsf{E} \subset \mathsf{K}$, we also have the relation $\langle \boldsymbol{u}_j,\, \boldsymbol{x} \rangle \leq 1$ for all $\boldsymbol{x} \in \mathsf{K}$. In particular, $\boldsymbol{u}_j \in \mathsf{E}^\circ$. Using the expression (14.3.2) for the polar ellipsoid, we see that

$$\sum\nolimits_{i=1}^{d} \lambda_i^2 \langle \boldsymbol{q}_i,\, \boldsymbol{u}_j \rangle^2 \leq 1.$$

The assumption in condition 2. yields

$$\sum\nolimits_{j=1}^{d} \alpha_j \sum\nolimits_{i=1}^{d} \lambda_i^2 \langle \boldsymbol{q}_i,\, \boldsymbol{u}_j \rangle^2 \leq \sum\nolimits_{j=1}^{m} \alpha_j = d.$$

Exchanging the order of the sums,

$$\sum\nolimits_{i=1}^{d} \lambda_i^2 \sum\nolimits_{j=1}^{m} \alpha_j \langle \boldsymbol{q}_i,\, \boldsymbol{u}_j \rangle^2 = \sum\nolimits_{i=1}^{d} \lambda_i^2 \|\boldsymbol{q}_i\|^2 = \sum\nolimits_{i=1}^{d} \lambda_i^2 \leq d.$$

The second relation is an easy consequence of the first equation in (14.5.1), and the last relation holds because the eigenvectors $\boldsymbol{q}_i$ have unit norm. By the AM–GM inequality,

$$\left( \prod\nolimits_{i=1}^{d} \lambda_i^2 \right)^{1/d} \leq \frac{1}{d} \sum\nolimits_{i=1}^{d} \lambda_i^2 \leq 1.$$

Moreover, all the inequalities hold with equality precisely when $\lambda_i^2 = 1$ for $i = 1, \dots, d$. This is the required result.

Next, we take up the more challenging proof that the first condition implies the second. Assume that $\mathsf{B}_d$ is the maximum-volume ellipsoid of $\mathsf{K}$. We need to show that there are contact points $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m \in \mathsf{B}_d \cap \operatorname{bd} \mathsf{K}$ and weights $\alpha_1, \dots, \alpha_m > 0$ that satisfy the condition (14.5.1).

It suffices to prove that

$$d^{-1} \mathbf{I}_d \in \operatorname{conv}\{\boldsymbol{u}\boldsymbol{u}^* : \boldsymbol{u} \in \mathsf{B}_d \cap \operatorname{bd} \mathsf{K}\} =: \mathsf{C} \subset \mathbb{H}_d.$$

We have written $\mathbb{H}_d$ for the real-linear space of $d \times d$ symmetric matrices, equipped with the trace inner product $\langle \boldsymbol{H},\, \boldsymbol{A} \rangle := \operatorname{trace}(\boldsymbol{H}\boldsymbol{A})$. The dimension of the linear space is $\binom{d+1}{2}$, so we can invoke Carathéodory's theorem to obtain a set of at most $m = \binom{d+1}{2} + 1$ contact points that satisfy the condition of the theorem.

To the contrary, assume that $d^{-1}\mathbf{I}_d \notin \mathsf{C}$. Since $\mathsf{B}_d \cap \operatorname{bd} \mathsf{K}$ is a compact set, the set $\mathsf{C}$ is compact and convex. Thus, there exists a linear functional on $\mathbb{H}_d$ that properly separates $d^{-1}\mathbf{I}$ from $\mathsf{C}$. In other terms, there is a matrix $\boldsymbol{H} \in \mathbb{H}_d$ such that

$$\langle \boldsymbol{H},\, d^{-1}\mathbf{I}_d \rangle < \langle \boldsymbol{H},\, \boldsymbol{u}\boldsymbol{u}^* \rangle \quad \text{for each } \boldsymbol{u} \in \mathsf{B}_d \cap \operatorname{bd} \mathsf{K}.$$

Equivalently,

$$d^{-1} \operatorname{trace} \boldsymbol{H} < \boldsymbol{u}^* \boldsymbol{H} \boldsymbol{u} \quad \text{for each } \boldsymbol{u} \in \mathsf{B}_d \cap \operatorname{bd} \mathsf{K}.$$

Since $\operatorname{trace}(d^{-1}\mathbf{I}_d) = 1 = \operatorname{trace}(\boldsymbol{u}\boldsymbol{u}^*)$, we can add a scaled identity matrix to $\boldsymbol{H}$ to ensure that $\operatorname{trace} \boldsymbol{H} = 0$ without affecting the separation inequalities. In summary,

$$0 < \boldsymbol{u}^* \boldsymbol{H} \boldsymbol{u} \quad \text{for each } \boldsymbol{u} \in \mathsf{B}_d \cap \operatorname{bd} \mathsf{K}.$$

From this condition, we will argue that it is possible to inscribe in $\mathsf{K}$ an ellipsoid with greater volume that $\mathsf{B}_d$.

For small $\delta > 0$, consider the ellipsoid

$$\mathsf{E}_\delta := \left\{ \boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^*(\mathbf{I} + \delta \boldsymbol{H})\boldsymbol{x} \leq 1 \right\}.$$

It is clear that the matrix $\mathbf{I} + \delta \boldsymbol{H}$ is positive definite, so this set is indeed an ellipsoid. Observe that $\mathsf{E}_\delta \neq \mathsf{B}_d$ for any $\delta > 0$. Moreover, no contact point $\boldsymbol{u} \in \mathsf{B}_d \cap \operatorname{bd} \mathsf{K}$ belongs to the ellipsoid $\mathsf{E}_\delta$:

$$\boldsymbol{u}^*(\mathbf{I} + \delta \boldsymbol{H})\boldsymbol{u} = 1 + \delta \boldsymbol{u}^* \boldsymbol{H} \boldsymbol{u} > 1.$$

It follows that the ellipsoid $\mathsf{E}_\delta$ does not intersect the boundary $\operatorname{bd} \mathsf{K}$ at *any* point because the boundary is compact. Therefore, $\mathsf{E}_\delta \subset \mathsf{K}$.

Now, let us show that $\operatorname{Vol}_d(\mathsf{E}_\delta) \geq \operatorname{Vol}_d(\mathsf{B}_d)$. To do so, we simply observe that

$$\det(\mathbf{I} + \delta \boldsymbol{H})^{1/d} \leq \frac{1}{d} \operatorname{trace}(\mathbf{I} + \delta \boldsymbol{H}) = 1 = \det(\mathbf{I}_d).$$

By the formula (14.3.1), we see that $\operatorname{Vol}_d(\mathsf{E}_\delta) \geq \operatorname{Vol}_d(\mathsf{B}_d)$. But the Euclidean ball is the *unique* maximum-volume ellipsoid, so $\mathsf{E}_\delta = \mathsf{B}_d$. This is a contradiction. □

## 14.6 Equivalence of Norms

As a simple application of John's characterization, let us prove one of the core results on the geometry of finite-dimensional Banach spaces.

**Theorem 14.6.1** (Equivalence of Norms)**.** *Let $\mathsf{K} \subset \mathbb{R}^d$ be a norm ball, and let $\mathsf{E} \subset \mathsf{K}$ be the maximum-volume inscribed ellipsoid. Then $\mathsf{K} \subset \sqrt{d}\,\mathsf{E}$.*

*Proof.* By affine covariance, we may assume that $\mathsf{K}$ is in John's position. Therefore, there are contact points $\boldsymbol{u}_i \in \mathsf{B}_d \cap \operatorname{bd} \mathsf{K}$ and weights $\alpha_i > 0$ that satisfy John's condition (14.5.1). Let $\boldsymbol{x} \in \mathsf{K}$. Lemma 14.5.2 implies that $\langle \boldsymbol{u}_i,\, \boldsymbol{x} \rangle \leq 1$ for each contact point $\boldsymbol{u}_i$. Therefore,

$$\|\boldsymbol{x}\|^2 = \sum\nolimits_{i=1}^m \alpha_i \langle \boldsymbol{u}_i,\, \boldsymbol{x} \rangle^2 \leq \sum\nolimits_{i=1}^d \alpha_i = d.$$

It follows that $\boldsymbol{x} \in \sqrt{d}\,\mathsf{B}_d$. This is the required result. □

For general culture, let us outline one of the main classical implications of this result. First, we define a notion of distance between pairs of norms on $\mathbb{R}^d$.

**Definition 14.6.2** (Banach–Mazur distance)**.** Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on $\mathbb{R}^d$. The Banach–Mazur "distance" between the norms is defined in terms of the associated operator norms:

$$\delta(\|\cdot\|_a, \|\cdot\|_b) := \inf\{\|\boldsymbol{T}\|_{a\to b}\|\boldsymbol{T}^{-1}\|_{b\to a} : \boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^d \text{ is nonsingular, linear}\}.$$

It can be shown that $\delta \geq 1$, with equality if and only if the two norms are equal. The function $\log \delta$ determines a metric on the family of norms on $\mathbb{R}^d$. With this metric, the norms on $\mathbb{R}^d$ compose a compact metric space, called the *Banach–Mazur compactum*.

Theorem 14.6.1 implies that every norm on $\mathbb{R}^d$ is within distance $\sqrt{d}$ of the standard Euclidean norm:

$$\delta(\|\cdot\|_a, \ell_2^d) \leq \sqrt{d}.$$

As a consequence, all pairs of norms on $\mathbb{R}^d$ lie within distance $d$ of each other:

$$\delta(\|\cdot\|_a, \|\cdot\|_b) \leq d.$$

It is, perhaps, surprising that

$$\delta(\ell_1^d, \ell_\infty^d) \leq \text{const} \cdot \sqrt{d}.$$

Nevertheless, in 1981, Gluskin proved that there are two norms that satisfy

$$\delta(\|\cdot\|_a, \|\cdot\|_b) \geq \text{const} \cdot d.$$

His proof constructs two realizations of a random polytope, and he shows that the two polytopes are very far apart.

# Lecture 15: The Reverse Isoperimetric Inequality

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*ACM 204, Fall 2018*
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*20 November 2018*

## 15.1 Agenda for Lecture 15

In this lecture, we develop a reversed isoperimetric inequality. This result identifies the convex body that has the *maximum* surface area at a given volume. To frame this question properly, we need consider an affine-invariant formulation. We approach the problem via Ball's theorem, which identifies the cube as the norm ball with largest volume relative to the volume of its inscribed ellipsoid. The proof of Ball's theorem is based on an analytic inequality, called the geometric Brascamp–Lieb inequality. In the last part of the lecture, we prove the geometric Brascamp–Lieb inequality using a mass transportation argument proposed by Barthe.

1. The reverse isoperimetric inequality
2. From Brascamp–Lieb to volume bounds
3. Brascamp–Lieb via mass transportation

## 15.2 The Reverse Isoperimetric Inequality

We first recall the isoperimetric inequality for convex bodies. For all convex bodies $\mathsf{C} \subset \mathbb{R}^d$,

$$\frac{S_{d-1}(\mathsf{C})}{\mathrm{Vol}_d(\mathsf{C})^{(d-1)/d}} \geq \frac{S_{d-1}(\mathsf{B}_d)}{\mathrm{Vol}_d(\mathsf{B}_d)^{(d-1)/d}}.$$

The left-hand side of this expression is called the isoperimetric quotient of $\mathsf{C}$, and it is interpreted as dimensionless ratio of surface area to volume.

The isoperimetric inequality states that, among all convex bodies, the Euclidean ball has the minimum isoperimetric quotient. It is natural to ask if there is a *reversed* isoperimetric inequality: among all convex bodies $\mathsf{C} \subset \mathbb{R}^d$, which one(s) have the *maximum* isoperimetric quotient? The question was initially raised in the 1930s. Unfortunately, this formulation is not well-posed because convex sets can be "flattened" such that their volume remains constant while their surface area grows unbounded, like a "pancake," so the isoperimetric quotient grows without bound.

To constrain the problem so that it is well posed, we only consider convex bodies in John's position. That is, we require that the maximum volume ellipsoid contained within the body be the Euclidean ball. Recall that for any convex body $\mathsf{C} \subset \mathbb{R}^d$ with $\mathrm{Vol}_d(\mathsf{C}) > 0$, there is an affine map $\boldsymbol{T}$ such that $\boldsymbol{T}(\mathsf{C})$ is in John's position. Our approach is to identify the maximum volume of a convex body in John's position. This problem was solved by Ball in 1991. For simplicity, we restrict our attention to norm balls in John's position.

**Theorem 15.2.1** (Ball, 1991)**.** *In $\mathbb{R}^d$, among norm balls in John's position, the cube*

$$\mathsf{Q}_d := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\ell_\infty} \leq 1\}$$

*has the maximum volume.*

We will prove Theorem 15.2.1 in Section 15.3. But first, let use state the reverse isoperimetric inequality formally and explain why it is a corollary of Ball's theorem.

**Corollary 15.2.2** (Reverse isoperimetric inequality)**.** *Every norm ball* $\mathsf{K} \subset \mathbb{R}^d$ *has an affine image* $\tilde{\mathsf{K}}$ *such that*

$$\mathrm{Vol}_d(\tilde{\mathsf{K}}) = \mathrm{Vol}_d(\mathsf{Q}_d) \quad \text{and} \quad S_{d-1}(\tilde{\mathsf{K}}) \leq S_{d-1}(\mathsf{Q}_d).$$

This statement can be rewritten in terms of isoperimetric quotients:

$$\inf_{\tilde{\mathsf{K}}} \frac{S_{d-1}(\tilde{\mathsf{K}})}{\mathrm{Vol}_d(\tilde{\mathsf{K}})^{(d-1)/d}} \leq \frac{S_{d-1}(\mathsf{Q}_d)}{\mathrm{Vol}_d(\mathsf{Q}_d)^{(d-1)/d}}.$$

The infimum ranges over all affine images of all norm balls $\mathsf{K} \subset \mathbb{R}^d$. This implies that, among all norm balls, the cube has the maximum surface area for a fixed volume once we apply an affine transform that minimizes surface area.

*Proof.* A cube $\mathsf{Q} \subset \mathbb{R}^d$ is a dilation of the $\ell_\infty$ ball $\mathsf{Q}_d$. First, let us compute the isoperimetric quotient of a cube $\mathsf{Q}$. We have

$$S_{d-1}(\mathsf{Q}) = 2d \cdot \mathrm{Vol}_d(\mathsf{Q})^{(d-1)/d}$$

because the cube has $2d$ facets and each facet is a cube of dimension $d-1$ with the same side length.

Let $\mathsf{K} \subset \mathbb{R}^d$ be a norm ball. To prove the theorem, we will find an affine transform $\tilde{\mathsf{K}}$ such that

$$S_{d-1}(\tilde{\mathsf{K}}) \leq 2d \cdot \mathrm{Vol}_d(\tilde{\mathsf{K}})^{(d-1)/d}. \tag{15.2.1}$$

To see why this is sufficient, observe that the cube $\mathsf{Q}$ with $\mathrm{Vol}_d(\mathsf{Q}) = \mathrm{Vol}_d(\tilde{\mathsf{K}})$ satisfies

$$S_{d-1}(\tilde{\mathsf{K}}) \leq 2d \cdot \mathrm{Vol}_d(\mathsf{Q})^{(d-1)/d} = S_{d-1}(\mathsf{Q}).$$

This is the reverse isoperimetric inequality.

Consider the affine image $\tilde{\mathsf{K}}$ of the norm ball that is in John's position; i.e., $\mathsf{B}_d$ is the maximum volume ellipsoid of $\tilde{\mathsf{K}}$. Then

$$\begin{aligned} S_{d-1}(\tilde{\mathsf{K}}) &= \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\tilde{\mathsf{K}} + \varepsilon \mathsf{B}_d) - \mathrm{Vol}_d(\tilde{\mathsf{K}})}{\varepsilon} \\ &\leq \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d((1+\varepsilon)\tilde{\mathsf{K}}) - \mathrm{Vol}_d(\tilde{\mathsf{K}})}{\varepsilon} \\ &= d \cdot \mathrm{Vol}_d(\tilde{\mathsf{K}}). \end{aligned}$$

The inequality holds because $\mathsf{B}_d \subset \tilde{\mathsf{K}}$, and we have used convexity to combine the two dilations of $\tilde{\mathsf{K}}$. Furthermore,

$$\begin{aligned} d \cdot \mathrm{Vol}_d(\tilde{\mathsf{K}}) &= d \cdot \mathrm{Vol}_d(\tilde{\mathsf{K}})^{1/d} \cdot \mathrm{Vol}_d(\tilde{\mathsf{K}})^{(d-1)/d} \\ &\leq d \cdot \mathrm{Vol}_d(\mathsf{Q}_d)^{1/d} \cdot \mathrm{Vol}_d(\tilde{\mathsf{K}})^{(d-1)/d} \\ &= 2d \cdot \mathrm{Vol}_d(\tilde{\mathsf{K}})^{(d-1)/d}. \end{aligned}$$

The first inequality follows from Ball's Theorem, and the final equality holds because $\mathrm{Vol}_d(\mathsf{Q}_d) = 2^d$. Combine the last two displays to obtain (15.2.1). This concludes the proof of the claim. □

Remark 15.2.3 (The reversed isoperimetric theorem for general convex bodies). If the symmetry assumption in Ball's Theorem is removed, then the regular simplex has the maximum volume among all convex bodies in John's position. Furthermore, the regular simplex solves the affine reverse isoperimetric problem among all convex bodies.

## 15.3 From Brascamp–Lieb to Volume Bounds

In this section, we explain how to derive Ball's Theorem from an important analytic inequality, called the geometric Brascamp–Lieb theorem. This result was established in a more general form by Brascamp & Lieb in 1976; Ball observed that their result simplifies dramatically if some of the parameters satisfy the condition that characterizes John's position.

**Theorem 15.3.1** (Brascamp & Lieb 1976; Ball 1991). *Consider unit vectors $\boldsymbol{u}_i \in \mathbb{R}^d$ and positive weights $\alpha_i > 0$ for $i = 1, \dots, m$ that satisfy John's condition:*

$$\sum\nolimits_{i=1}^{m} \alpha_i \, \boldsymbol{u}_i \boldsymbol{u}_i^* = \mathbf{I}_d \quad \textit{and} \quad \sum\nolimits_{i=1}^{m} \alpha_i = d. \tag{15.3.1}$$

*For all nonnegative, integrable functions $f_i : \mathbb{R} \to \mathbb{R}$,*

$$\int_{\mathbb{R}^d} \prod\nolimits_{i=1}^{m} f_i(\langle \boldsymbol{u}_i, \, \boldsymbol{x} \rangle)^{\alpha_i} \, \mathrm{d}\boldsymbol{x} \leq \prod\nolimits_{i=1}^{m} \left( \int_{\mathbb{R}} f_i(t) \, \mathrm{d}t \right)^{\alpha_i}. \tag{15.3.2}$$

*Equality is achieved when each $f_i$ is the same Gaussian function, such as*

$$f_i(t) = \mathrm{e}^{-\pi t^2} \quad \textit{for } i = 1, \dots, m.$$

The geometric Brascamp–Lieb inequality is very powerful. It contains familiar inequalities, including the Hölder inequality, sharp Young inequalities for convolution, and a version of the Loomis–Whitney projection inequality. There is also a reversed version, discussed on the homework, that contains the Prékopa–Leindler inequality. We will prove Theorem 15.3.1 in the next section.

The geometric Brascamp–Lieb inequality is well suited for proving results about cubes. Indeed, we can take the functions $f_i$ to be indicators of the unit interval ($f_i = \mathbb{1}_{[-1,+1]}$). Then the product on the right-hand side simplifies to the volume of a cube. As an example of this type, let us show how Theorem 15.3.1 implies Ball's theorem.

*Proof of Theorem 15.2.1.* Let $\mathsf{K} \subset \mathbb{R}^d$ be a norm ball, and assume that $\mathsf{K}$ is in John's Position. We need to show that

$$\mathrm{Vol}_d(\mathsf{K}) \leq \mathrm{Vol}_d(\mathsf{Q}_d) = 2^d.$$

To do so, we use John's theorem to construct an outer approximation of $\mathsf{K}$ whose volume is easier to compute.

Since $\mathsf{B}_d \subset \mathsf{K}$, by John's theorem, there are symmetric contact points $\pm \boldsymbol{u}_i \in \mathsf{B} \cap \mathrm{bd}\, \mathsf{K}$ and positive weights $\alpha_i > 0$ for $i = 1, \dots, m$ that satisfy (15.3.1). We define a bounded polyhedron $\mathsf{P}$ via

$$\mathsf{P} := \{ \boldsymbol{x} \in \mathbb{R}^d : |\langle \boldsymbol{u}_i, \, \boldsymbol{x} \rangle| \leq 1 \text{ for } i = 1, \dots, m \}.$$

Note that $\mathsf{K} \subset \mathsf{P}$, as shown in Figure 15.1. Define the functions $f_i = \mathbb{1}_{[-1,+1]}$, and observe that the indicator of the polyhedron takes the form

$$\mathbb{1}_{\mathsf{P}}(\boldsymbol{x}) = \prod\nolimits_{i=1}^{m} \mathbb{1}_{[-1,+1]}(\langle \boldsymbol{u}_i, \, \boldsymbol{x} \rangle) = \prod\nolimits_{i=1}^{m} \mathbb{1}_{[-1,+1]}(\langle \boldsymbol{u}_i, \, \boldsymbol{x} \rangle)^{\alpha_i}.$$

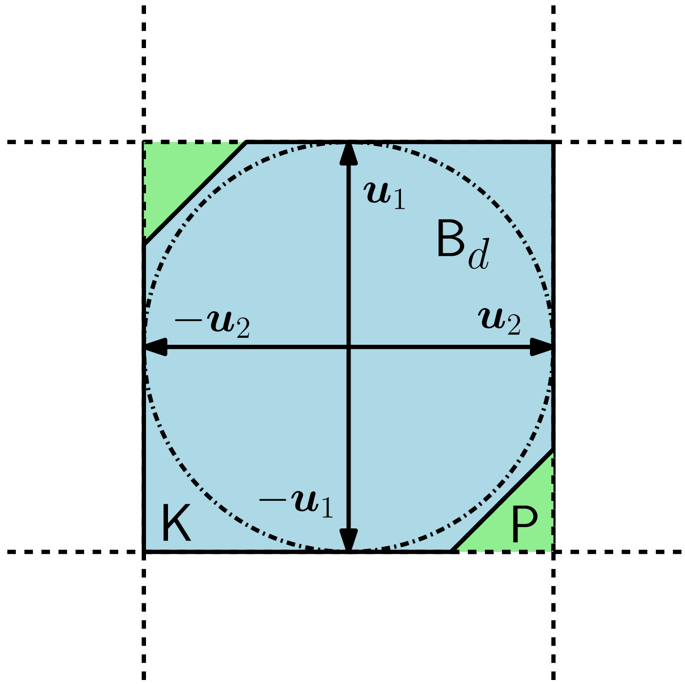


**Figure 15.1** (Proof of Ball's theorem). This figure illustrates a norm ball $\mathsf{K}$ (blue) in John's position. The Euclidean ball $\mathsf{B}_d$ is the maximum-volume inscribed ellipsoid. The contact points $\pm\boldsymbol{u}_i$ between $\mathsf{K}$ and $\mathsf{B}_d$ are marked with arrows. The polyhedron $\mathsf{P}$ (blue and green) is the intersection of the negative halfspaces that support the convex body $\mathsf{K}$ at the contact points $\pm\boldsymbol{u}_i$.

Note that we can introduce the power of $\alpha_i$ because the indicator function only assumes the values 0 and 1.

We can use this expression to bound the volume of the set $\mathsf{K}$. As $\mathsf{K} \subset \mathsf{P}$, we see that

$$\begin{aligned}\mathrm{Vol}_d(\mathsf{K}) \leq \mathrm{Vol}_d(\mathsf{P}) &= \int_{\mathbb{R}^d} \mathbb{1}_{\mathsf{P}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d} \prod\nolimits_{i=1}^m \mathbb{1}_{[-1,+1]}(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^{\alpha_i}\,\mathrm{d}\boldsymbol{x} \\ &\leq \prod\nolimits_{i=1}^m \left( \int_{\mathbb{R}^d} \mathbb{1}_{[-1,+1]}(t)\,\mathrm{d}t \right)^{\alpha_i} = \prod\nolimits_{i=1}^m 2^{\alpha_i} = 2^d.\end{aligned}$$

The second inequality is (15.3.2). At the last step, we have used the second part of John's condition. This calculation yields the desired fact that $\mathrm{Vol}_d(\mathsf{K}) \leq 2^d$. □

## 15.4 Brascamp–Lieb via Mass Transportation

We will establish Theorem 15.3.1 using a mass transportation argument developed by Franck Barthe (1998). Extensions of this approach yield the equality conditions, have higher-dimensional extensions, and imply the reversed geometric Brascamp–Lieb inequality.

### 15.4.1 Consequences of John's Condition

First, let us state a lemma that extracts the information we need from John's condition (15.3.1).

**Lemma 15.4.1** (Consequences of John's condition)**.** *The assumption* (15.3.1) *implies the following inequalities.*

1. *For all scalars* $\theta_i$*, if* $\boldsymbol{y} = \sum_{i=1}^m \alpha_i \theta_i \, \boldsymbol{u}_i$*, then* $\|\boldsymbol{y}\|^2 \leq \sum_{i=1}^m \alpha_i \theta_i^2$*.*
2. *For each linear map* $\boldsymbol{T}$ *on* $\mathbb{R}^d$*,*
$$\det(\boldsymbol{T}) \leq \prod\nolimits_{i=1}^m \|\boldsymbol{T}\boldsymbol{u}_i\|^{\alpha_i}.$$
3. *For all scalars* $\theta_i > 0$*,*
$$\det\Big(\sum\nolimits_{i=1}^m \theta_i \alpha_i \, \boldsymbol{u}_i \boldsymbol{u}_i^*\Big) \geq \prod\nolimits_{i=1}^m \theta_i^{\alpha_i}.$$

The proof of the third statement in the lemma requires a variational representation of the determinant. This type of formula is sometimes called *quasi-linearization* in the matrix analysis literature.

**Fact 15.4.2** (Variational representation of determinant)**.** *Let* $\boldsymbol{A} \in \mathbb{R}^{d \times d}$ *be a positive-definite matrix. Then*
$$(\det \boldsymbol{A})^{1/d} = \min\bigg\{ \frac{1}{d} \operatorname{trace}(\boldsymbol{T}\boldsymbol{A}) : \det \boldsymbol{T} = 1 \text{ and } \boldsymbol{T} \text{ is positive definite} \bigg\}.$$

For completeness, we will establish this fact in a moment. But first, let us prove Lemma 15.4.1.

*Proof of Lemma 15.4.1.* To establish the first statement, define the matrix $\boldsymbol{U}$ with columns $\boldsymbol{u}_i$, the diagonal matrix $\boldsymbol{A}^2 = \operatorname{diag}(\alpha_1, \dots, \alpha_m)$, and the vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m)$. The usual operator norm inequality implies

$$\|\boldsymbol{y}\|^2 = \|\boldsymbol{U}\boldsymbol{A}^2\boldsymbol{\theta}\|^2 \leq \|\boldsymbol{U}\boldsymbol{A}\|^2 \|\boldsymbol{A}\boldsymbol{\theta}\|^2.$$

The first term equals 1:

$$\|\boldsymbol{U}\boldsymbol{A}\|^2 = \|\boldsymbol{U}\boldsymbol{A}^2\boldsymbol{U}^*\| = \Big\|\sum\nolimits_{i=1}^m \alpha_i \, \boldsymbol{u}_i \boldsymbol{u}_i^*\Big\| = \|\mathbf{I}_d\| = 1.$$

The second term can be expanded as

$$\|\boldsymbol{A}\boldsymbol{\theta}\|^2 = \sum\nolimits_{i=1}^m \alpha_i \theta_i^2.$$

Combine the last three displays to obtain the result.

To obtain the second result, we may assume that $\boldsymbol{T}$ is positive semidefinite without loss of generality. Let $\boldsymbol{T} = \sum_{j=1}^d \lambda_i \boldsymbol{q}_j \boldsymbol{q}_j^*$ be an eigenvalue decomposition. Then

$$\|\boldsymbol{T}\boldsymbol{u}_i\|^2 = \sum\nolimits_{j=1}^d \lambda_j^2 \langle \boldsymbol{q}_j, \, \boldsymbol{u}_i \rangle^2 \geq \prod\nolimits_{j=1}^d \lambda_j^{2\langle \boldsymbol{q}_j, \, \boldsymbol{u}_i \rangle^2}.$$

We have used the AM–GM inequality, as a privilege of the fact $\sum_j \langle \boldsymbol{q}_j, \boldsymbol{u}_i \rangle^2 = 1$, which holds because the eigenvectors compose an orthonormal basis. It follows that

$$\begin{aligned}\prod_{i=1}^m \|\boldsymbol{T}\boldsymbol{u}_i\|^{\alpha_i} &\geq \prod_{i=1}^m \prod_{j=1}^d \lambda_j^{\alpha_i \langle \boldsymbol{q}_j, \boldsymbol{u}_i^2 \rangle} \\ &= \prod_{j=1}^d \lambda_j^{\sum_{i=1}^m \alpha_i \langle \boldsymbol{q}_j, \boldsymbol{u}_i \rangle^2} = \prod_{j=1}^d \lambda_j = \det \boldsymbol{T}.\end{aligned}$$

We have used John's condition (15.3.1) at the second-to-last step, along with the fact that the eigenvectors have unit norm.

Last, we dualize the second result to obtain the third inequality. According to Fact 15.4.2, there is a matrix $\boldsymbol{T}^*\boldsymbol{T}$ with determinant 1 that satisfies

$$\begin{aligned}\det\left(\sum_{i=1}^m \theta_i \alpha_i \, \boldsymbol{u}_i \boldsymbol{u}_i^*\right)^{1/d} &= \frac{1}{d} \operatorname{trace}\left(\sum_{i=1}^m \theta_i \alpha_i \, (\boldsymbol{T}\boldsymbol{u}_i)(\boldsymbol{T}\boldsymbol{u}_i)^*\right) \\ &= \sum_{i=1}^m (\alpha_i/d) \, \theta_i \, \|\boldsymbol{T}\boldsymbol{u}_i\|^2 \\ &\geq \left(\prod_{i=1}^m \theta_i^{\alpha_i}\right)^{1/d} \left(\prod_{i=1}^m \|\boldsymbol{T}\boldsymbol{u}_i\|^{\alpha_i}\right)^{2/d} \geq \left(\prod_{i=1}^m \theta_i^{\alpha_i}\right)^{1/d}.\end{aligned}$$

The first inequality is AM–GM, which is valid since $\sum_{i=1}^m \alpha_i = d$. The last inequality follows from part 2 of the lemma. □

We conclude this section with a proof of the variational representation of the determinant.

*Proof of Fact 15.4.2.* It is an exercise to argue that the minimum is achieved. By changing coordinates, we may assume that the minimizer is a diagonal matrix. Parameterize $\boldsymbol{T} = \operatorname{diag}(t_{11}, \dots, t_{dd})$. Taking the logarithm of the determinant constraint, the minimization problem is

$$\text{minimize} \quad \frac{1}{d} \sum_{i=1}^d t_{ii} a_{ii} \quad \text{subject to} \quad \sum_{i=1}^d \log t_{ii} = 0.$$

This optimization problem can be written in a convex form (by relaxing the equality constraint to an inequality), so the minimum is achieved when the KKT conditions are in force. Introducing a Lagrange multiplier $\beta$, we have the relations

$$d^{-1} a_{ii} = \beta t_{ii}^{-1} \quad \text{for } i = 1, \dots, d.$$

To satisfy the equality constraint, we must have

$$0 = \sum_{i=1}^d \log t_{ii} = \sum_{i=1}^d \log(\beta d a_{ii}^{-1}) = d \log(\beta d) - \log\left(\prod_{i=1}^d a_{ii}\right).$$

Using the last two displays in sequence,

$$\frac{1}{d} \sum_{i=1}^d t_{ii} a_{ii} = \beta d = \left(\prod_{i=1}^d a_{ii}\right)^{1/d}.$$

Invoke Hadamard's determinant inequality to arrive at the bound

$$\frac{1}{d} \operatorname{trace}(\boldsymbol{T}\boldsymbol{A}) = \frac{1}{d} \sum_{i=1}^d t_{ii} a_{ii} = \left(\prod_{i=1}^d a_{ii}\right)^{1/d} \geq (\det \boldsymbol{A})^{1/d}.$$

Equality holds when $\boldsymbol{A}$ is diagonal. This completes the argument.

Finally, let us summarize an easy proof of Hadamard's determinant inequality. Let $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{C}^*$ be a Cholesky decomposition. Then

$$\det \boldsymbol{A} = \det(\boldsymbol{C}\boldsymbol{C}^*) = (\det \boldsymbol{C})^2 = \left(\prod\nolimits_{i=1}^d c_{ii}\right)^2 \leq \prod\nolimits_{i=1}^d a_{ii}.$$

Indeed, the construction of the Cholesky decomposition ensures that $c_{ii}^2 \leq a_{ii}$. □

### 15.4.2 Proof of the Geometric Brascamp–Lieb Inequality

First, let us verify that Gaussian functions achieve equality in Theorem 15.3.1. Define $f_i(t) = \mathrm{e}^{-\pi t^2}$ for each index $i$. Since $\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t = 1$, the right-hand side of (15.3.2) equals one. As for the left-hand side,

$$\begin{aligned}\int_{\mathbb{R}^d} \prod\nolimits_{i=1}^m f_i(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^{\alpha_i}\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^d} \prod\nolimits_{i=1}^m \mathrm{e}^{-\pi\alpha_i\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle^2}\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{-\pi\sum_{i=1}^m \alpha_i\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle^2}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \mathrm{e}^{-\pi\|\boldsymbol{x}\|^2}\,\mathrm{d}\boldsymbol{x} = 1.\end{aligned}$$

The penultimate identity is an immediate consequence of John's condition (15.3.1). This demonstrates that Gaussian functions saturate the inequality (15.3.2).

Next, we establish the inequality (15.3.2). To that end, we may assume that each function $f_i$ is positive, continuous, and has integral 1. We wish to show that

$$I := \int_{\mathbb{R}^d} \prod\nolimits_{i=1}^m f_i(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^{\alpha_i}\,\mathrm{d}\boldsymbol{x} \leq 1.$$

To do so, we transport the mass of a Gaussian function (which saturates the inequality) to the functions $f_i$ that we are actually given. We will show that this operation decreases the value of the integral from the value it attains for Gaussians.

Let $g(a) := \mathrm{e}^{-\pi a^2}$ be the Gaussian function whose integral equals 1. For each $i$, construct the optimal transport map $T_i : \mathbb{R} \to \mathbb{R}$ via the equation

$$\int_{-\infty}^{T_i(a)} g(x)\,\mathrm{d}x = \int_{-\infty}^{a} f_i(x)\,\mathrm{d}x.$$

Since $f_i$ and $g$ are positive and continuous, $T_i$ is increasing and differentiable. It follows from elementary calculus that

$$g(T_i(a)) \cdot T_i'(a) = f_i(a) \quad \text{for all } a \in \mathbb{R}.$$

We can substitute these expressions into the integral $I$ to obtain

$$\begin{aligned} I &= \int_{\mathbb{R}^d} \prod\nolimits_{i=1}^m f_i(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^{\alpha_i}\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d} \left(\prod\nolimits_{i=1}^m g(T_i(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle))^{\alpha_i}\right)\left(\prod\nolimits_{i=1}^m T_i'(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^{\alpha_i}\right)\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{-\pi\sum_{i=1}^m \alpha_i T_i(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^2}\left(\prod\nolimits_{i=1}^m T_i'(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^{\alpha_i}\right)\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{15.4.1}$$

We must see how this expression relates to the integral of a Gaussian.

Make the inspired change of variables

$$\boldsymbol{y} = \sum\nolimits_{i=1}^{m} \alpha_i T_i(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)\, \boldsymbol{u}_i.$$

According to Lemma 15.4.1(1),

$$\|\boldsymbol{y}\|^2 \leq \sum\nolimits_{i=1}^{m} \alpha_i T_i(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^2.$$

The Jacobian of the change of variables is easily computed as

$$J(\boldsymbol{y}, \boldsymbol{x}) = \det\Big(\sum\nolimits_{i=1}^{m} T_i'(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle) \cdot \alpha_i\, \boldsymbol{u}_i \boldsymbol{u}_i^*\Big) \geq \prod\nolimits_{i=1}^{m} T_i'(\langle \boldsymbol{u}_i,\, \boldsymbol{x}\rangle)^{\alpha_i}.$$

The inequality follows from Lemma 15.4.1(3). Substitute the last two displays into the expression (15.4.1) for the integral to arrive at

$$I \leq \int_{\mathbb{R}^d} \mathrm{e}^{-\pi \|\boldsymbol{y}\|^2}\, J(\boldsymbol{y}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \mathrm{e}^{-\pi \|\boldsymbol{y}\|^2}\, \mathrm{d}\boldsymbol{y} = 1.$$

The second relation is just the standard change of variables formula from multidimensional calculus.

# Lecture 16: Mixed Volumes

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*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*27 November 2018*

## 16.1 Agenda for Lecture 16

For a convex body $\mathsf{C}$ in $\mathbb{R}^d$, the parallel body at level $\lambda \geq 0$ is the set $\mathsf{C} + \lambda \mathsf{B}_d$. In Lecture 8, we established Steiner's formula, which states that the function $\lambda \mapsto \mathrm{Vol}_d(\mathsf{C} + \lambda \mathsf{B}_d)$ is a polynomial for $\lambda \geq 0$. The coefficients appearing in this polynomial are called *intrinsic volumes*, and we spent some time developing their properties. Over the course of Lectures 9–12, we saw time and time again that intrinsic volumes are truly fundamental objects in convex geometry.

Today, we introduce a substantial generalization of intrinsic volumes. Where before we considered the univariate polynomial $\lambda \mapsto \mathrm{Vol}_d(\mathsf{C} + \lambda \mathsf{B}_d)$, we will now consider multivariate functions of the form

$$(\lambda_1, \dots, \lambda_m) \mapsto \mathrm{Vol}_d(\lambda_1 \mathsf{C}_1 + \cdots + \lambda_m \mathsf{C}_m) \tag{16.1.1}$$

induced by a collection of $m$ convex bodies $\mathsf{C}_i$ in $\mathbb{R}^d$. Much like the volume of a parallel body, the function (16.1.1) is a (multivariate) polynomial in the parameters $\boldsymbol{\lambda} \geq \mathbf{0}$. For a family of convex bodies, $\{\mathsf{C}_i\}_{i \in [m]}$, the coefficients of (16.1.1) are called the *mixed volumes* of $\{\mathsf{C}_i\}_{i \in [m]}$. Mixed volumes are also fundamental quantities that capture many features of a family of convex bodies.

The primary purpose of this lecture is to prove that (16.1.1) is indeed a polynomial in $\boldsymbol{\lambda}$. This result is called Minkowski's theorem. We will also cover some basic properties of mixed volumes, although the real exploration of mixed volumes begins in the next lecture. Here is the order of business:

1. Support functions: The key to deriving Minkowski's theorem
2. Minkowski's theorem on mixed volumes
3. Mixed volumes and intrinsic volumes

## 16.2 Support Functions: The Key to Deriving Minkowski's Theorem

Our derivation of Minkowski's theorem on mixed volumes involves different tools from the proof of Steiner's formula in Lecture 8. As usual, we establish the result for a family $\mathsf{C}_1, \dots, \mathsf{C}_m$ of polytopes, and we extend the result to other convex bodies by approximation. See Figure 16.1 for an illustration of how weighted Minkowski sums behave with polytopes. Whereas we proved Steiner's Formula by evaluating a high-dimensional integral in polar coordinates, our proof of Minkowski's theorem on mixed volumes proceeds by induction. Support functions play a crucial role in the inductive step.

Recall that the *support function* of a convex set $\mathsf{C} \subset \mathbb{R}^d$ is the function $h_{\mathsf{C}}$ defined by

$$h_{\mathsf{C}}(\boldsymbol{u}) = \sup\{\langle \boldsymbol{u}, \boldsymbol{x} \rangle : \boldsymbol{x} \in \mathsf{C}\} \quad \text{for } \boldsymbol{u} \in \mathbb{R}^d.$$

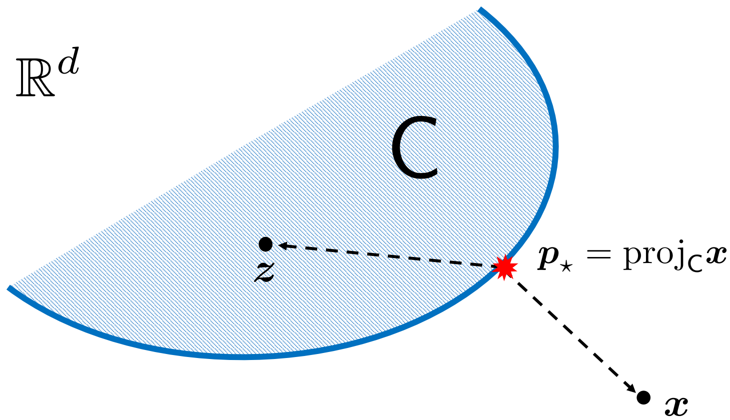


**Figure 16.1** (Weighted Minkowski sums of two polygons). The figure illustrates $\mu\mathsf{C} + \lambda\mathsf{K}$ for a handful of values of $\mu, \lambda \geq 0$. When $\mu = \lambda > 0$ we obtain regular octogons, labeled "S." If $\mu \gg \lambda$ then we encounter shapes resembling $\tilde{\mathsf{C}}$, while if $\lambda \gg \mu$ the result is a shape more like $\tilde{\mathsf{K}}$. When both weights are positive, the resulting polygon always has eight faces. Observe that weighted Minkowski sums of polytopes do not exhibit the "rounded corners" that occur with parallel bodies.

In this lecture, we typically restrict our attention to the value of the support funciton on the unit sphere. A simple but fundamental property of the support function is linearity in the set-argument. That is, for convex bodies $\mathsf{C}$ and $\mathsf{K}$ and nonnegative scalars $\mu, \lambda$, we have

$$h_{\mu\mathsf{C}+\lambda\mathsf{K}} = \mu h_{\mathsf{C}} + \lambda h_{\mathsf{K}}.$$

This result appeared on your homework.

Next, let us introduce the set-valued mapping that returns the face exposed in the direction $\boldsymbol{u}$:

$$\mathsf{F}_{\mathsf{C}}(\boldsymbol{u}) = \arg\max\{\langle \boldsymbol{u}, \boldsymbol{x}\rangle : \boldsymbol{x} \in \mathsf{C}\} \quad \text{for } \boldsymbol{u} \in \mathbb{R}^d.$$

This mapping is well-defined because convex bodies are closed and bounded. In particular, note that

$$\mathsf{F}_{\mathsf{C}}(\mathbf{0}) = \mathsf{C}.$$

It may come as a surprise that the face map $\mathsf{F}_{\mathsf{C}}$ enjoys the same type of linearity property as the support function.

**Lemma 16.2.1.** *Let $\mathsf{C}_1, \dots, \mathsf{C}_m$ be fixed convex bodies in $\mathbb{R}^d$. For a nonnegative vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$, define $\mathsf{C}(\boldsymbol{\lambda}) = \lambda_1\mathsf{C}_1 + \cdots + \lambda_m\mathsf{C}_m$. Then*

$$\mathsf{F}_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u}) = \sum\nolimits_{i=1}^{m} \lambda_i \mathsf{F}_{\mathsf{C}_i}(\boldsymbol{u}) \quad \textit{for each } \boldsymbol{u} \in \mathbb{R}^d.$$

*In this expression, the summation sign refers to a Minkowski sum.*

*Proof.* The result is trivial when $\boldsymbol{u} = \mathbf{0}$. It suffices to establish the result for a fixed unit vector $\boldsymbol{u}$.

First, we show that any point $\boldsymbol{x} \in \mathsf{F}_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u})$ can be expressed as a $\boldsymbol{\lambda}$-weighted sum of elements from $\mathsf{F}_{\mathsf{C}_i}(\boldsymbol{u})$. Since $\mathsf{F}_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u})$ is contained in $\mathsf{C}(\boldsymbol{\lambda})$, we know that $\boldsymbol{x}$ can be written as $\boldsymbol{x} = \sum_{i=1}^m \lambda_i \boldsymbol{x}_i$ for some $\boldsymbol{x}_i \in \mathsf{C}_i$. Now, using $\langle \boldsymbol{u}, \boldsymbol{x}_i \rangle \leq h_{\mathsf{C}_i}(\boldsymbol{u})$, we can expand the inner product $\langle \boldsymbol{u}, \boldsymbol{x} \rangle = h_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u})$ to obtain

$$h_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{x} \rangle = \sum_{i=1}^m \lambda_i \langle \boldsymbol{u}, \boldsymbol{x}_i \rangle \leq \sum_{i=1}^m \lambda_i h_{\mathsf{C}_i}(\boldsymbol{u}) = \sum_{i=1}^m h_{\lambda_i \mathsf{C}_i}(\boldsymbol{u}) = h_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u}).$$

The final equality follows from the additivity of support functions. From the preceding equation, it is clear that $\lambda_i \langle \boldsymbol{u}, \boldsymbol{x}_i \rangle < \lambda_i h_{\mathsf{C}_i}(\boldsymbol{u})$ *cannot happen*. Thus, when $\lambda_i > 0$, we have $\langle \boldsymbol{u}, \boldsymbol{x}_i \rangle = h_{\mathsf{C}_i}(\boldsymbol{u})$, and so $\boldsymbol{x}_i \in \mathsf{F}_{\mathsf{C}_i}(\boldsymbol{u})$. When $\lambda_i = 0$, we can simply replace $\boldsymbol{x}_i$ by an arbitrary element of $\mathsf{F}_{\mathsf{C}_i}(\boldsymbol{u})$ while still maintaining the condition $\boldsymbol{x} = \sum_{i=1}^m \lambda_i \boldsymbol{x}_i$.

To complete the proof, we must verify that, if $\boldsymbol{x}_i \in \mathsf{F}_{\mathsf{C}_i}(\boldsymbol{u})$, then the point $\boldsymbol{x} := \sum_{i=1}^m \lambda_i \boldsymbol{x}_i$ belongs to $\mathsf{F}_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u})$. Indeed,

$$\langle \boldsymbol{u}, \boldsymbol{x} \rangle = \sum_{i=1}^m \lambda_i \langle \boldsymbol{u}, \boldsymbol{x}_i \rangle = \sum_{i=1}^m \lambda_i h_{\mathsf{C}_i}(\boldsymbol{u}) = h_{\sum_{i=1}^m \lambda_i \mathsf{C}_i}(\boldsymbol{u}) = h_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u}).$$

Since $\boldsymbol{x}$ evidently belongs to $\mathsf{C}(\boldsymbol{\lambda})$, the last display ensures that $\boldsymbol{x} \in \mathsf{F}_{\mathsf{C}(\boldsymbol{\lambda})}(\boldsymbol{u})$. □

Now, linearity of $\mathsf{F}_{\mathsf{C}}$ is all well and good, but how does it get us any closer to deriving a polynomial expression for the volume of a weighted Minkowski sum? The answer lays in the following lemma, which gives an expression for the volume of a polytope $\mathsf{P}$ in terms of the support function $h_{\mathsf{P}}$ and the face map $\mathsf{F}_{\mathsf{P}}$. This formula will support an induction on the dimension of the polytope.

**Lemma 16.2.2.** *Let $\mathsf{P} \subset \mathbb{R}^d$ be a polytope with a nonempty interior, and let $\mathscr{U}$ be the (finite) set of outer unit normals to the facets of $\mathsf{P}$. Then*

$$\mathrm{Vol}_d(\mathsf{P}) = \frac{1}{d} \sum_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{P}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})).$$

*Proof sketch.* This result was on the homework, so we give only a brief outline of the argument.

First, consider the case where $\mathbf{0}$ belongs to the polytope $\mathsf{P}$. For every facet $\mathsf{H}$ of $\mathsf{P}$, there is some unit vector $\boldsymbol{u}$ in $\mathscr{U}$ for which $\mathsf{F}_{\mathsf{P}}(\boldsymbol{u}) = \mathsf{H}$. Using cone for the conic hull operator, we can write

$$\mathsf{P} = \bigcup_{\boldsymbol{u} \in \mathscr{U}} \mathrm{cone}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})) \cap \{\boldsymbol{x} : \langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq h_{\mathsf{P}}(\boldsymbol{u})\}. \tag{16.2.1}$$

Each term appearing in the union is a pyramid[1] with base $\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})$ and height $h_{\mathsf{P}}(\boldsymbol{u})$. Thus, by applying the result from Homework 2 Problem 2(b), we have

$$\mathrm{Vol}_d\left(\mathrm{cone}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})) \cap \{\boldsymbol{x} : \langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq h_{\mathsf{P}}(\boldsymbol{u})\}\right) = \frac{h_{\mathsf{P}}(\boldsymbol{u})}{d} \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})).$$

To complete the argument, one need only verify that (1) distinct terms in the union from Equation 16.2.1 have disjoint relative interiors, and (2) that any vector $\boldsymbol{u} \in \mathscr{U}$ where $\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})$ is *not* a facet of $\mathsf{P}$ necessarily contributes no mass to the union.

[1]In the sense of Homework 2, Problem 2.

If the polytope does not contain the origin, we use a translation argument. If $\boldsymbol{x} \in \mathsf{P}$, then the first part of the proof implies that

$$\begin{aligned} \mathrm{Vol}_d(\mathsf{P}) = \mathrm{Vol}_d(\mathsf{P} - \boldsymbol{x}) &= \frac{1}{d} \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{P}-\boldsymbol{x}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}-\boldsymbol{x}}(\boldsymbol{u})) \\ &= \frac{1}{d} \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{P}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})) - \frac{1}{d} \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} \langle \boldsymbol{u}, \boldsymbol{x} \rangle \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})). \end{aligned}$$

Translation does not change the directions that expose facets, nor the shape of the facets. We have also used additivity of support functions. Finally, observe that the last sum equals zero, as you showed in Problem Set 2, Problem 2. □

At this point, a derivation of the Minkowski theorem is beginning to take shape. If we set $\mathsf{P} = \sum_{i=1}^m \lambda_i \mathsf{P}_i$ for polytopes $\mathsf{P}_i$ and nonnegative scalars $\lambda_i$, then for an appropriate index set $\mathscr{U}$ we could simultaneously apply Lemmas 16.2.1 and 16.2.2 to write

$$\mathrm{Vol}_d(\mathsf{P}) = \frac{1}{d} \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} (\lambda_1 h_{\mathsf{P}_1}(\boldsymbol{u}) + \cdots + \lambda_m h_{\mathsf{P}_m}(\boldsymbol{u})) \cdot \mathrm{Vol}_{d-1}(\lambda_1 \mathsf{F}_{\mathsf{P}_1}(\boldsymbol{u}) + \cdots \lambda_m \mathsf{F}_{\mathsf{P}_m}(\boldsymbol{u})).$$

We would then appeal to an appropriate induction hypothesis to establish that the $\mathrm{Vol}_{d-1}(\cdots)$ term is a polynomial.

The preceding outline is indeed the main idea in our proof of Minkowski's theorem on mixed volumes. However, one issue remains: the set $\mathscr{U}$ of directions was chosen as a function of $\mathsf{P}$, but $\mathsf{P}$ is a function of $\boldsymbol{\lambda}$. To prove that $\boldsymbol{\lambda} \mapsto \mathrm{Vol}_d(\sum_{i=1}^m \lambda_i \mathsf{P}_i)$ is a polynomial, we would also need to make sure that the set $\mathscr{U}$ does not depend on $\boldsymbol{\lambda}$. Luckily for us, this is true.

**Lemma 16.2.3.** *Fix nonempty polytopes $\mathsf{P}_1, \dots, \mathsf{P}_m \subset \mathbb{R}^d$. For $\boldsymbol{\lambda} \in \mathbb{R}_+^m$, define the polytope $\mathsf{P}(\boldsymbol{\lambda}) := \sum_{i=1}^m \lambda_i \mathsf{P}_i$. Let $\mathscr{U}$ be the (finite) set of outer unit normals to the facets of $\mathsf{P}(1)$. Then the outer unit normals of the facets of $\mathsf{P}(\boldsymbol{\lambda})$ all appear in $\mathscr{U}$.*

*Proof.* It suffices to show that the affine hull of any facet of $\mathsf{P}(\boldsymbol{\lambda})$ is a translate of the affine hull of a facet of $\mathsf{P}(\mathbf{1})$.

For each $i \in [m]$, pick a face $\mathsf{F}_i$ from $\mathsf{P}_i$, and let $\boldsymbol{\lambda} \in \mathbb{R}_+^m$ be such that $\mathsf{F}(\boldsymbol{\lambda}) := \sum_{i=1}^m \lambda_i \mathsf{F}_i$ is a facet of $\mathsf{P}(\boldsymbol{\lambda})$. By translation, we can assume that each $\mathrm{relint}(\mathsf{F}_i)$ contains the origin.[2] Now, let $\mathsf{H}$ be the hyperplane containing $\mathsf{F}(\boldsymbol{\lambda})$. Since it contains the origin, $\mathsf{H}$ is the *linear* hull of $\mathsf{F}(\boldsymbol{\lambda})$. Without loss of generality, we can assume $\lambda_i > 0$. Therefore,

$$\mathsf{H} = \sum\nolimits_{i=1}^m \mathrm{lin}(\lambda_i \mathsf{F}_i) = \sum\nolimits_{i=1}^m \mathrm{lin}(\mathsf{F}_i) = \mathrm{lin}\Big(\sum\nolimits_{i=1}^m \mathsf{F}_i\Big) = \mathrm{lin}(\mathsf{F}(\mathbf{1})).$$

Thus, we see that $\mathrm{aff}(\mathsf{F}(\boldsymbol{\lambda})) = \mathrm{aff}(\mathsf{F}(\mathbf{1})) = \mathsf{H}$.

Thus we have shown that $\mathsf{F}(\mathbf{1})$ is parallel to $\mathsf{F}(\boldsymbol{\lambda})$ and has dimension $d-1$. To confirm that $\mathsf{F}(\mathbf{1})$ is a facet of $\mathsf{P}(\mathbf{1})$, we invoke Lemma 16.2.1. □

[2] Note that any translation of the $\mathsf{F}_i$ will not affect whether or not $\mathsf{F}(\boldsymbol{\lambda})$ and $\mathsf{F}(\mathbf{1})$ are in parallel hyperplanes.

## 16.3 Minkowski's Theorem on Mixed Volumes

We are now prepared to state and prove Minkowski's theorem on mixed volumes.

**Theorem 16.3.1** (Minkowski)**.** *There is a permutation-invariant function $V : (\mathscr{C}_d)^d \to \mathbb{R}$ on a family of $d$ convex bodies in $\mathbb{R}^d$ that satisfies*

$$\mathrm{Vol}_d(\lambda_1 \mathsf{C}_1 + \cdots + \lambda_m \mathsf{C}_m) = \sum\nolimits_{i_1,\ldots,i_d=1}^{m} V(\mathsf{C}_{i_1}, \ldots, \mathsf{C}_{i_d}) \cdot \lambda_{i_1} \cdots \lambda_{i_d} \tag{16.3.1}$$

*for all convex bodies $\{\mathsf{C}_j\}_{j=1}^m \subset \mathscr{C}_d$ and all $\boldsymbol{\lambda} \in \mathbb{R}_+^m$. The function $V$ is called the* mixed volume*.*

We focus on the case when each $\mathsf{C}_j = \mathsf{P}_j$ is a polytope in $\mathbb{R}^d$. The general result follows from the fact that $\mathrm{Vol}_d$ and weighted Minkowski summation are Hausdorff continuous. We also rely on the fact that the homogeneous polynomials of fixed degree are closed under limits.

*Proof.* The proof is by induction on the dimension $d$. When $d = 1$ all convex bodies are line segments and the claim is trivial. Henceforth consider $d > 1$, and assume the theorem up to dimension $d - 1$. In the argument, we will notate the mixed volume in dimension $d$ as $V_{[d]}$ for clarity.

Let $\mathscr{U}$ denote the (finite) set of outer unit-normals of facets of $\sum_{j=1}^m \mathsf{P}_j$. By Lemma 16.2.3, $\mathscr{U}$ contains the outer unit normals of every facet of $\mathsf{P}(\boldsymbol{\lambda}) := \sum_{j=1}^m \lambda_j \mathsf{P}_j$, regardless of the choice of $\boldsymbol{\lambda} \in \mathbb{R}_+^m$.

In order to keep subsequent expressions legible, for each $(j, \boldsymbol{u})$ in $[m] \times \mathscr{U}$ we abbreviate $\mathsf{F}_j^{\boldsymbol{u}} := \mathsf{F}_{\mathsf{P}_j}(\boldsymbol{u})$ and $h_j^{\boldsymbol{u}} := h_{\mathsf{P}_j}(\boldsymbol{u})$. Using these new symbols, Lemmas 16.2.1 and 16.2.2 give us

$$\mathrm{Vol}_d(\mathsf{P}(\boldsymbol{\lambda})) = \frac{1}{d} \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} (\lambda_1 h_1^{\boldsymbol{u}} + \cdots + \lambda_m h_m^{\boldsymbol{u}}) \cdot \mathrm{Vol}_{d-1}(\lambda_1 \mathsf{F}_1^{\boldsymbol{u}} + \cdots + \lambda_m \mathsf{F}_m^{\boldsymbol{u}}) \tag{16.3.2}$$

where $\lambda_1 \mathsf{F}_1^{\boldsymbol{u}} + \cdots + \lambda_m \mathsf{F}_m^{\boldsymbol{u}} =: \mathsf{F}_{\boldsymbol{\lambda}}^{\boldsymbol{u}}$ is the face of $\mathsf{P}(\boldsymbol{\lambda})$ exposed by $\boldsymbol{u}$.

Now, we prepare for the inductive step. The induction hypothesis states that there exists a permutation-invariant function $V_{[d-1]} : \mathscr{C}_{d-1}^{d-1} \to \mathbb{R}$ satisfying

$$\mathrm{Vol}_{d-1}(\lambda_1 \mathsf{Q}_1 + \cdots \lambda_m \mathsf{Q}_m) = \sum\nolimits_{\boldsymbol{i} \in [m]^{d-1}} V_{[d-1]}(\mathsf{Q}_{i_1}, \ldots, \mathsf{Q}_{i_{d-1}}) \cdot \lambda_{i_1} \cdots \lambda_{i_{d-1}} \tag{16.3.3}$$

for every collection of polytopes $\mathsf{Q}_1, \ldots, \mathsf{Q}_m \subset \mathscr{C}_{d-1}$. This statement does not immediately help us reduce Equation 16.3.2 because the argument to $\mathrm{Vol}_{d-1}$ consists of convex bodies in $\mathbb{R}^d$. To resolve this, we need to extend $V_{[d-1]}$ to $(d-1)$-tuples of convex bodies in $\mathbb{R}^d$ that are contained in parallel hyperplanes (orthogonal to the direction $\boldsymbol{u} \in \mathscr{U}$). In this case, we can determine the value of $V_{[d-1]}$ by translating all of the lower-dimensional convex bodies into the parallel hyperplane containing the origin, and we can compute the mixed volume there by means of the inductive hypothesis.

Having thus extended $V_{[d-1]}$, we apply the induction hypothesis for fixed direction $\boldsymbol{u}$ to write

$$\mathrm{Vol}_{d-1}(\lambda_1 \mathsf{F}_1^{\boldsymbol{u}} + \cdots \lambda_m \mathsf{F}_m^{\boldsymbol{u}}) = \sum\nolimits_{\boldsymbol{i} \in [m]^{d-1}} V_{[d-1]}(\mathsf{F}_{i_1}^{\boldsymbol{u}}, \ldots, \mathsf{F}_{i_{d-1}}^{\boldsymbol{u}}) \cdot \lambda_{i_1} \cdots \lambda_{i_{d-1}}.$$

Substitute this expression into Equation 16.3.2 to reach

$$\mathrm{Vol}_d(\mathsf{P}(\boldsymbol{\lambda})) = \frac{1}{d}\sum\nolimits_{\boldsymbol{u}\in\mathscr{U}}\sum\nolimits_{\boldsymbol{i}\in[m]^{d-1}} V_{[d-1]}(\mathsf{F}^{\boldsymbol{u}}_{i_1},\dots,\mathsf{F}^{\boldsymbol{u}}_{i_{d-1}})\cdot\lambda_{i_1}\cdots\lambda_{i_{d-1}}\Big[\sum\nolimits_{j=1}^{m}\lambda_j h^{\boldsymbol{u}}_j\Big].$$

Now, set $j\in[m]$ to be the $d$th coordinate of a *new* multiindex $\boldsymbol{i}\in[m]^d$ that we form by appending $j$ to the original multiindex $\boldsymbol{i}\in[m]^{d-1}$. This allows us to write

$$\begin{aligned}
\mathrm{Vol}_d(\mathsf{P}(\boldsymbol{\lambda})) &= \frac{1}{d}\sum\nolimits_{\boldsymbol{u}\in\mathscr{U}}\sum\nolimits_{\boldsymbol{i}\in[m]^{d}} h^{\boldsymbol{u}}_{i_d} V_{[d-1]}(\mathsf{F}^{\boldsymbol{u}}_{i_1},\dots,\mathsf{F}^{\boldsymbol{u}}_{i_{d-1}})\cdot\lambda_{i_1}\cdots\lambda_{i_d}\\
&= \frac{1}{d}\sum\nolimits_{\boldsymbol{i}\in[m]^{d}}\sum\nolimits_{\boldsymbol{u}\in\mathscr{U}} h^{\boldsymbol{u}}_{i_d} V_{[d-1]}(F^{\boldsymbol{u}}_{i_1},\dots,F^{\boldsymbol{u}}_{i_{d-1}})\cdot\lambda_{i_1}\cdots\lambda_{i_d}\\
&= \sum\nolimits_{\boldsymbol{i}\in[m]^{d}}\lambda_{i_1}\cdots\lambda_{i_d}\sum\nolimits_{\boldsymbol{u}\in\mathscr{U}} h^{\boldsymbol{u}}_{i_d} V_{[d-1]}(F^{\boldsymbol{u}}_{i_1},\dots,F^{\boldsymbol{u}}_{i_{d-1}})/d. \qquad (16.3.4)
\end{aligned}$$

We are nearly done. In view of (16.3.4), we might want to define

$$\hat{V}(\mathsf{P}_{i_1},\dots,\mathsf{P}_{i_d}) := \sum\nolimits_{\boldsymbol{u}\in\mathscr{U}} h^{\boldsymbol{u}}_{i_d} V_{[d-1]}(\mathsf{F}^{\boldsymbol{u}}_{i_1},\dots,\mathsf{F}^{\boldsymbol{u}}_{i_{d-1}})/d. \qquad (16.3.5)$$

This expression is not evidently invariant under interchanges of distinct $\mathsf{P}_{i_d}$ and $\mathsf{P}_{i_j}$. To resolve this, we construct $V$ by symmetrizing the $\hat{V}$ over all permutations of $[d]$; that is,

$$V(\mathsf{P}_{i_1},\dots,\mathsf{P}_{i_d}) := \frac{1}{d!}\sum\nolimits_{\sigma\in\mathsf{Sym}_d}\hat{V}(\mathsf{P}_{i_{\sigma(1)}},\dots,\mathsf{P}_{i_{\sigma(d)}}).$$

This completes our proof. □

## 16.4 Mixed Volumes and Intrinsic Volumes

To begin to understand mixed volumes, one may refer to a two-dimensional illustration; see Figure 16.2.

We may also wonder whether there is any connection between mixed volumes and the intrinsic volumes that have taken so much of our attention. In fact, the intrinsic volumes are specific cases of the mixed volumes. Apply Steiner's formula and the $m=2$ case of Minkowski's theorem. Then group monomials to obtain

$$\begin{aligned}
\mathrm{Vol}_d(\mathsf{C}+\lambda\mathsf{B}_d) &= \sum\nolimits_{j=0}^{d}\lambda^{d-j}1^j\binom{d}{j}V(\underbrace{\mathsf{C},\dots,\mathsf{C}}_{j\text{ times}},\underbrace{\mathsf{B}_d,\dots,\mathsf{B}_d}_{d-j\text{ times}})\\
&= \sum\nolimits_{j=0}^{d}\lambda^{d-j}\kappa_{d-j}V_j(\mathsf{C}).
\end{aligned}$$

By comparing coefficients, we find that

$$V_j(\mathsf{C}) = \frac{1}{\kappa_{d-j}}\binom{d}{j}V(\underbrace{\mathsf{C},\dots,\mathsf{C}}_{j\text{ times}},\underbrace{\mathsf{B}_d,\dots,\mathsf{B}_d}_{d-j\text{ times}}).$$

In the next lecture, we will continue our discussion of mixed volumes, and we will develop some of their main properties.

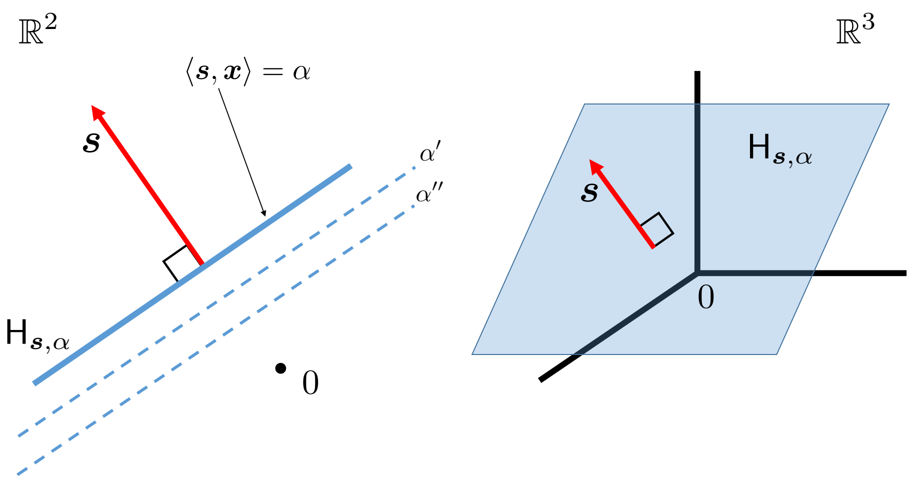


**Figure 16.2** (Mixed volumes in two dimensions). For the two convex bodies $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^2$, the area of $\mathsf{C} + \mathsf{K}$ can be clearly divided into four regions: the area of $\mathsf{C}$, the area of $\mathsf{K}$, and the area of the shaded region along with its reflection. This symmetry of $\mathsf{C} + \mathsf{K}$ makes it seem perfectly natural that $V(\mathsf{C}, \mathsf{K}) = V(\mathsf{K}, \mathsf{C})$. Note that $V$ is symmetric because we *make it so*; there are choices of $\mathsf{C}, \mathsf{K}$ for which $[\mathsf{C} + \mathsf{K}] \setminus [\mathsf{C} \cup \mathsf{K}]$ does not exhibit any meaningful symmetry.

# Lecture 17: Properties of Mixed Volumes

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*ACM 204, Fall 2018*
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*29 November 2018*

## 17.1 Agenda for Lecture 17

In the last lecture we introduced mixed volumes, which can be seen as generalizations of intrinsic volumes that are defined in terms of mulitple convex bodies. After reminding ourselves of last lecture's main result, we will explore additional examples and prove basic properties of mixed volumes.

1. Mixed volumes, recap from last lecture
2. Examples of mixed volumes
3. Basic properties of mixed volumes
4. Monotonicity of mixed volumes

## 17.2 Mixed Volumes and Minkowski's Theorem

Let $\mathscr{C}_d$ denote the space of convex bodies in $\mathbb{R}^d$. In the last lecture, we proved Minkowski's theorem on mixed volumes.

**Theorem 17.2.1** (Minkowski)**.** *There exists a permutation-invariant map* $V : (\mathscr{C}_d)^d \longrightarrow \mathbb{R}$*, such that for all* $\mathsf{C}_1, \dots, \mathsf{C}_m \in \mathscr{C}_d$ *and* $\lambda_1, \dots, \lambda_m \geq 0$*, we have*

$$\mathrm{Vol}_d(\lambda_1 \mathsf{C}_1 + \dots + \lambda_m \mathsf{C}_m) = \sum\nolimits_{i_1, \dots, i_d = 1}^{m} V(\mathsf{C}_{i_1}, \dots, \mathsf{C}_{i_d}) \cdot \lambda_{i_1} \cdots \lambda_{i_d}$$

Thus, the volume of the Minkowski sum $\lambda_1 \mathsf{C}_1 + \dots + \lambda_m \mathsf{C}_m$ is a degree-$d$ homogeneous polynomial in the $m$ scalar variables $\lambda_1, \dots, \lambda_m$. The coefficients of the polynomial are given in terms of the function $V$, evaluated at different combinations of $d$ of the sets $\mathsf{C}_1, \dots, \mathsf{C}_m$, chosen with repetition.

**Definition 17.2.2** (Mixed volumes)**.** We call the function $V$ from Theorem 17.2.1 the *mixed volume.* For $\mathsf{C}_1, \dots, \mathsf{C}_d \in \mathscr{C}_d$, we call $V(\mathsf{C}_1, \dots, \mathsf{C}_d)$ the *mixed volume of* the family $(\mathsf{C}_1, \dots, \mathsf{C}_d)$.

By considering the case $m = d$ in Minkowski's theorem, we derive the following alternative expression for the mixed volume:

**Lemma 17.2.3** (Derivative formulation of mixed volumes)**.** *We have the following alternative expression for the mixed volume:*

$$d! \cdot V(\mathsf{C}_1, \dots \mathsf{C}_d) = \frac{\partial^d}{\partial \lambda_1 \cdots \partial \lambda_d} \mathrm{Vol}_d(\lambda_1 \mathsf{C}_1 + \dots + \lambda_d \mathsf{C}_d)$$

*Proof.* This expression follows when we apply Theorem 17.2.1 and observe that summands where the indices $i_1, \dots, i_d$ are distinct have mixed derivative equal to the mixed volume, while all other summands have a zero mixed derivative. □

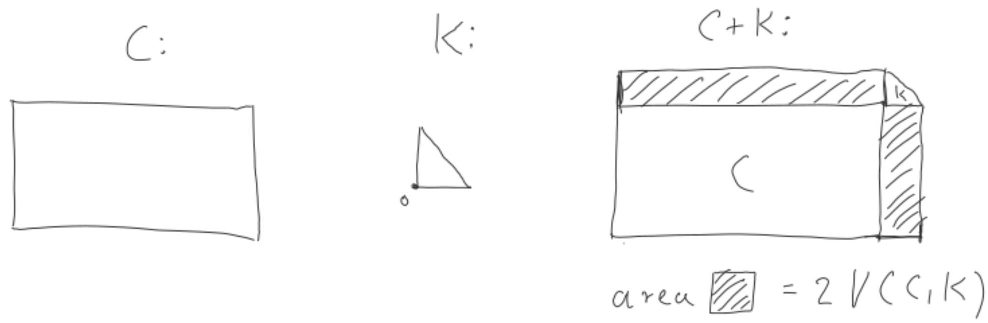


**Figure 17.1** (Two-dimensional mixed volumes). An instance of mixed volumes in two dimensions; see formula (17.3.1).

## 17.3 Examples of Mixed Volumes

To help us understand the scope of the mixed volumes, we consider a collection of geometric examples.

**Example 17.3.1** (The two-dimensional case)**.** In $\mathbb{R}^2$, we have

$$2V(\mathsf{C}, \mathsf{K}) = \mathrm{Vol}_2(\mathsf{C} + \mathsf{K}) - \mathrm{Vol}_2(\mathsf{C}) - \mathrm{Vol}_2(\mathsf{K}). \tag{17.3.1}$$

This identity is illustrated in Figure 17.1.

When we include only two convex bodies, the formula in Theorem 17.2.1 simplifies dramatically.

$$\mathrm{Vol}_d(\lambda\mathsf{C} + \mu\mathsf{K}) = \sum_{i=0}^{d} \binom{d}{i} \lambda^i \mu^{d-i} \cdot V(\underbrace{\mathsf{C}, \dots, \mathsf{C}}_{i \text{ times}}, \underbrace{\mathsf{K}, \dots, \mathsf{K}}_{d-i \text{ times}}) \tag{17.3.2}$$

The mixed volumes of two bodies already describe a number of interesting geometric settings.

**Example 17.3.2** (Mixed volumes with a line segment)**.** Let $\mathsf{C} \subset \mathbb{R}^d$ be a convex body. Write $\mathsf{C} \,|\, \mathsf{L}$ for the orthogonal projection of $\mathsf{C}$ onto a subspace $\mathsf{L}$. Consider a line segment $\mathsf{U} := [\mathbf{0}, \boldsymbol{u}] \subset \mathbb{R}^d$. We then have

$$\begin{aligned}\mathrm{Vol}_d(\lambda\mathsf{C} + \mu\mathsf{U}) &= \mathrm{Vol}_d(\lambda\mathsf{C}) + \mathrm{length}(\mu\mathsf{U}) \cdot \mathrm{Vol}_d\big(\lambda\mathsf{C} \,\big|\, \boldsymbol{u}^\perp\big) \\ &= \lambda^d\, \mathrm{Vol}_d(\mathsf{C}) + \lambda^{d-1}\mu\, \mathrm{length}(\mathsf{U}) \cdot \mathrm{Vol}_d\big(\mathsf{C} \,|\, \boldsymbol{u}^\perp\big).\end{aligned}$$

Thus, by matching coefficients with (17.3.2), we get

$$\begin{aligned} V(\mathsf{C}, \dots, \mathsf{C}) &= \mathrm{Vol}_d(\mathsf{C}) \quad \text{and} \\ V(\mathsf{C}, \dots, \mathsf{C}, \mathsf{U}) &= \frac{1}{d}\, \mathrm{length}(\mathsf{U}) \cdot \mathrm{Vol}_d\big(\mathsf{C} \,\big|\, \boldsymbol{u}^\perp\big).\end{aligned}$$

All the other mixed volumes are zero. This is illustrated in Figure 17.2.

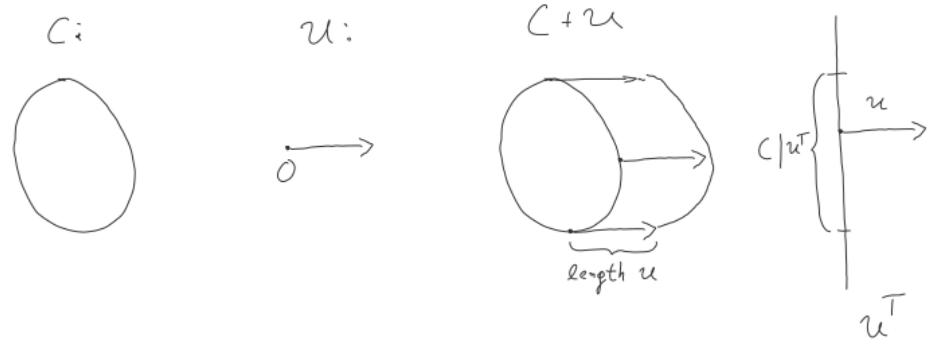


**Figure 17.2** (Mixed volumes with a segment). This picture illustrates the formula derived in Example 17.3.2 for the mixed volumes of a convex body $\mathsf{C}$ and a line segment $\mathsf{U} = [\mathbf{0}, \boldsymbol{u}]$.

In Example 17.3.2, for a unit vector $\boldsymbol{u}$, we have seen that $d \cdot V(\mathsf{C}, \dots, \mathsf{C}, [\mathbf{0}, \boldsymbol{u}])$ gives us the volume of the projection of $\mathsf{C}$ onto the hyperplane $\boldsymbol{u}^{\perp}$. Indeed, there are more instances where mixed volumes in $\mathsf{C}$ are related to the volumes of projections of $\mathsf{C}$.

**Example 17.3.3** (Quermassintegrals)**.** In the last lecture, we obtained the following relationship between mixed volumes and intrinsic volumes.

$$\binom{d}{i} V(\underbrace{\mathsf{C}, \dots, \mathsf{C}}_{i \text{ times}}, \underbrace{\mathsf{B}_d, \dots, \mathsf{B}_d}_{d-i \text{ times}}) = \kappa_{d-i} V_i(\mathsf{C}). \tag{17.3.3}$$

Furthermore, Kubota showed that intrinsic volumes are related to the mean values of projections as follows.

$$V_i(\mathsf{C}) = \frac{\kappa_d}{\kappa_i \kappa_{d-i}} \binom{d}{i} \int_{G(i,d)} V_i(\mathsf{C} \,|\, \mathsf{L}) \, \mathrm{d}\nu_j(\mathsf{L}).$$

Here, $\nu_i$ denotes the rotation-invariant probability measure on the Grassmannian $G(i,d)$, which consists of $i$-dimensional subspaces in $\mathbb{R}^d$. As usual, $\mathsf{C} \,|\, \mathsf{L}$ denotes the projection of $\mathsf{C}$ onto $\mathsf{L}$. Combining these equations, we obtain

$$W_{d-i}^{(d)}(\mathsf{C}) := V(\underbrace{\mathsf{C}, \dots, \mathsf{C}}_{i \text{ times}}, \underbrace{\mathsf{B}_d, \dots, \mathsf{B}_d}_{d-i \text{ times}}) = \frac{\kappa_d}{\kappa_i} \int_{G(i,d)} \mathrm{Vol}_i(\mathsf{C} \,|\, \mathsf{L}) \, \mathrm{d}\nu_j(\mathsf{L}).$$

In words, the mixed volume is proportional to the mean $i$-dimensional volume of the projections of $\mathsf{C}$ onto the $i$-dimensional subspaces.

The quantity $W_i^{(d)}$ is called the $i$th *quermassintegral* of the set $\mathsf{C}$. This German word translates, roughly, as mean projection volume. The quermassintegrals arise naturally from the definition of mixed volumes, and they are equivalent to intrinsic volumes, modulo scaling. Quermassintegrals depend on the dimension $d$, in contrast to the intrinsic volumes, which do not.

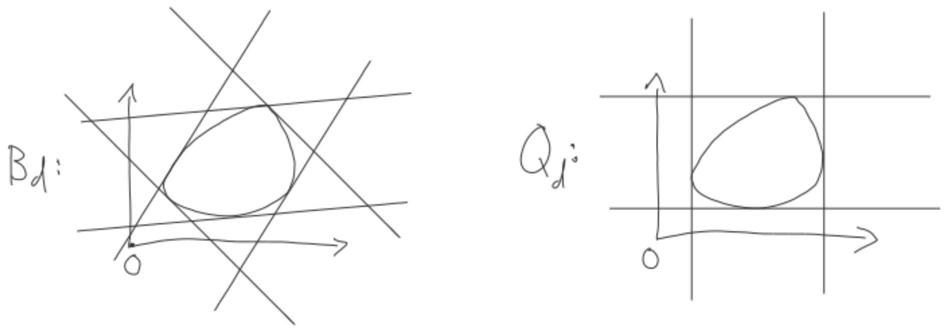


**Figure 17.3** (Mean projection volumes). [left] The mixed volume $V(\mathsf{C}, \mathsf{B}_d, \dots, \mathsf{B}_d)$ is proportional to the average volume of the projections of $\mathsf{C}$ onto all hyperplanes. [right] The mixed volume $V(\mathsf{C}, \mathsf{Q}_d, \dots, \mathsf{Q}_d)$ is proportional to the average size of projections of $\mathsf{C}$ onto hyperplanes spanned by coordinate axes.

Next, we present an analogue of Example 17.3.3 where balls are replaced by cubes.

**Example 17.3.4** (Coordinate projection volumes)**.** Let $\mathsf{Q}_d$ denote the $\ell_\infty^d$ unit ball. In his 1984 doctoral thesis, Pajor showed that

$$\binom{d}{i} V(\underbrace{\mathsf{C}, \dots, \mathsf{C}}_{i \text{ times}}, \underbrace{\mathsf{Q}_d, \dots, \mathsf{Q}_d}_{d-i \text{ times}}) = \frac{\mathrm{Vol}_d(\mathsf{Q}_d)}{\mathrm{Vol}_i(\mathsf{Q}_i)} \sum\nolimits_{|I|=i} \mathrm{Vol}_i(\boldsymbol{P}_I \mathsf{C}),$$

The sum extends over sets $I \subset \{1, \dots, d\}$, and $\boldsymbol{P}_I$ denotes the orthogonal projection from $\mathbb{R}^d$ onto $\mathbb{R}^I$. Thus, we have

$$V(\underbrace{\mathsf{C}, \dots, \mathsf{C}}_{i \text{ times}}, \underbrace{\mathsf{Q}_d, \dots, \mathsf{Q}_d}_{d-i \text{ times}}) = 2^{d-i} \binom{d}{i}^{-1} \sum\nolimits_{|I|=i} \mathrm{Vol}_i(\boldsymbol{P}_I \mathsf{C}).$$

The mixed volume is proportional to the mean volume of $i$-dimensional projections *along the coordinate axes*. This formula was derived independently by Leinster (2012) in a different geometric setting.

We conclude this section with a list of miscellaneous related examples.

**Example 17.3.5** (More mixed volumes)**.** Let $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$ be convex bodies, and write $\diamond_d$ for the $\ell_1^d$ unit ball.

1. $V(\mathsf{C}, \dots, \mathsf{C}, \mathsf{B}_d)$ is proportional to the Minkowski surface area of $\mathsf{C}$.
2. $V(\mathsf{C}, \dots, \mathsf{C}, \mathsf{Q}_d)$ is proportional to the mean $(d-1)$-volume of the projections of $\mathsf{C}$ onto the coordinate hyperplanes.
3. $V(\mathsf{C}, \dots, \mathsf{C}, \mathsf{K})$ is proportional to the $\mathsf{K}$-surface area of the set $\mathsf{C}$.
4. $V(\mathsf{C}, \mathsf{B}_d, \dots, \mathsf{B}_d)$ is proportional to the mean width of $\mathsf{C}$.

5. $V(\mathsf{C}, \mathsf{Q}_d, \dots, \mathsf{Q}_d)$ is proportional to the mean length of the projections of $\mathsf{C}$ onto individual coordinates.
6. $V(\mathsf{C}, \diamond_d, \dots, \diamond_d)$ is proportional to the Rademacher width of $\mathsf{C}$:

$$V(\mathsf{C}, \diamond_d, \dots, \diamond_d) \propto \mathbb{E}_{\boldsymbol{\varepsilon}} \max_{\boldsymbol{x} \in \mathsf{C}} \langle \boldsymbol{\varepsilon}, \boldsymbol{x} \rangle$$

where $\boldsymbol{\varepsilon} \sim \textsc{uniform}\{\pm 1\}^d$.

This last result also appears in Pajor's thesis.

## 17.4 Basic Properties of Mixed Volumes

The main goal of this lecture is to outline the main facts about mixed volumes. The first result is an alternative formula for the mixed volumes that shows how they depend on the volumes of the summands.

**Proposition 17.4.1** (Inclusion–Exclusion)**.** *For $\mathsf{C}_1, \dots, \mathsf{C}_d \in \mathscr{C}_d$, we have*

$$\begin{aligned} d! \cdot V(\mathsf{C}_1, \dots \mathsf{C}_d) &= \sum\nolimits_{k=1}^{d} (-1)^{n+k} \sum\nolimits_{i_1 < \dots < i_k} \mathrm{Vol}_d(\mathsf{C}_{i_1} + \dots + \mathsf{C}_{i_k}) \\ &= \mathrm{Vol}_d(\mathsf{C}_1, \dots, \mathsf{C}_d) - [\mathrm{Vol}_d(\mathsf{C}_1 + \cdots \mathsf{C}_{d-1}) + \dots + \mathrm{Vol}_d(\mathsf{C}_2 + \dots + \mathsf{C}_d)] \\ &\quad + \dots + (-1)^{d-1} [\mathrm{Vol}_d(\mathsf{C}_1) + \dots + \mathrm{Vol}_d(\mathsf{C}_d)]. \end{aligned}$$

*Proof sketch.* Denote the right hand side of the above equation by $f(\mathsf{C}_1, \dots, \mathsf{C}_d)$. By Theorem 17.2.1, the function $(\lambda_1, \cdots, \lambda_d) \mapsto f(\lambda_1 \mathsf{C}_1, \dots, \lambda_d \mathsf{C}_d)$ is a degree-$d$ polynomial in the variables $\lambda_1, \dots, \lambda_d$. We will show that every term in $f(\lambda_1 \mathsf{C}_1, \dots, \lambda_d \mathsf{C}_d)$ vanishes, except for the term with monomial $\lambda_1 \cdots \lambda_d$. Indeed, we have

$$\begin{aligned} &(-1)^{n+1} f(\{\mathbf{0}\}, \mathsf{C}_2, \dots, \mathsf{C}_d) \\ &\quad = \sum\nolimits_{2 \le i \le d} \mathrm{Vol}_d(\mathsf{C}_i) - \left[ \sum\nolimits_{2 \le j \le d} \mathrm{Vol}_d(\{\mathbf{0}\} + \mathsf{C}_j) + \sum\nolimits_{2 \le i < j \le d} \mathrm{Vol}_d(\mathsf{C}_i + \mathsf{C}_j) \right] \\ &\qquad + \left[ \sum\nolimits_{2 \le j < k \le d} \mathrm{Vol}_d(\{\mathbf{0}\} + \mathsf{C}_j + \mathsf{C}_k) + \sum_{2 \le i < j < k \le d} \mathrm{Vol}_d(\mathsf{C}_i + \mathsf{C}_j + \mathsf{C}_k) \right] - \cdots \\ &\quad = 0. \end{aligned}$$

By repeating the above argument with $\lambda_i = 0$ for each $i$, we obtain

$$f(\lambda_1 \mathsf{C}_1, \dots, \lambda_d \mathsf{C}_d) \propto \lambda_1 \cdots \lambda_d.$$

By Theorem 17.2.1, this implies the result. □

Next, we note that mixed volumes reproduce the volume as a special case. (Conversely, the mixed volumes can be obtained by polarization of the volume.)

**Proposition 17.4.2** (Volume)**.** *For all $\mathsf{C} \in \mathscr{C}_d$,*

$$V(\mathsf{C}, \dots, \mathsf{C}) = \mathrm{Vol}_d(\mathsf{C}). \tag{17.4.1}$$

*Proof.* This point follows from Equation (17.3.3), with $i = d$. □

Like the volume, the mixed volumes interact nicely with isometries of Euclidean space and with affine maps.

**Proposition 17.4.3** (Affine motions). *The mixed volume is invariant under translation; that is for all* $\mathsf{C}_1, \ldots, \mathsf{C}_d \in \mathscr{C}_d$ *and* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d \in \mathbb{R}^d$*, we have*

$$V(\mathsf{C}_1 + \boldsymbol{x}_1, \ldots, \mathsf{C}_d + \boldsymbol{x}_d) = V(\mathsf{C}_1, \ldots, \mathsf{C}_d).$$

*The mixed volume is affine covariant. For all* $\boldsymbol{T} \in \mathbb{R}^{d\times d}$*, we have*

$$V(\boldsymbol{T}\mathsf{C}_1, \ldots, \boldsymbol{T}\mathsf{C}_d) = |\det \boldsymbol{T}| \cdot V(\mathsf{C}_1, \ldots, \mathsf{C}_d).$$

*In particular, if* $\boldsymbol{T}$ *is a rigid motion, the mixed volume is unchanged.*

*Proof.* The results follow from the translation invariance and affine covariance of the volume, respectively. □

The mixed volume is Minkowski additive in each coordinate.

**Proposition 17.4.4** (Linearity). *The mixed volume is Minkowski additive in each coordinate. For* $\alpha, \beta \geq 0$ *and* $\mathsf{C}, \mathsf{K}, \mathsf{D}_2, \ldots, \mathsf{D}_d \in \mathscr{C}_d$*, we have*

$$V(\alpha\mathsf{C} + \beta\mathsf{K}, \mathsf{D}_2, \ldots, \mathsf{D}_d) = \alpha V(\mathsf{C}, \mathsf{D}_2, \ldots, \mathsf{D}_d) + \beta V(\mathsf{K}, \mathsf{D}_2, \ldots, \mathsf{D}_d).$$

*Proof.* We have

$$\begin{aligned}\mathrm{Vol}_d(\lambda_1(\alpha\mathsf{C} + \beta\mathsf{K}) + \lambda_2\mathsf{D}_2 + \cdots + \lambda_d\mathsf{D}_d)& \\ &= \mathrm{Vol}_d(\lambda_1\alpha\mathsf{C} + \lambda_1\beta\mathsf{K} + \lambda_2\mathsf{D}_2 + \cdots + \lambda_d\mathsf{D}_d).\end{aligned}$$

We expand both sides of the above equation in terms of Theorem 17.2.1. The $\lambda_1 \cdots \lambda_d$ monomial on the left-hand side is given by $d! \cdot V(\alpha\mathsf{C} + \beta\mathsf{K}, \mathsf{D}_2, \ldots, \mathsf{D}_d)$, while the one of the right-hand side is given by $d! \cdot \alpha V(\mathsf{C}, \mathsf{D}_2, \ldots, \mathsf{D}_d) + d! \cdot \beta V(\mathsf{K}, \mathsf{D}_2, \ldots, \mathsf{D}_d)$, from which we obtain the result. □

Like the volume, the mixed volume is also a valuation in each coordinate.

**Proposition 17.4.5** (Valuation). *The mixed volume is a set valuation with respect to each coordinate. That is, for all* $\mathsf{D}_2, \ldots \mathsf{D}_d \in \mathscr{C}_d$ *the map*

$$\mathsf{K} \mapsto V(\mathsf{K}, \mathsf{D}_2, \ldots, \mathsf{D}_d)$$

*is a set valuation.*

*Proof.* We know that $\mathrm{Vol}_d(\cdot + \mathsf{K})$ is a set valuation. For $\mathsf{C}, \mathsf{K}, \mathsf{C} \cup \mathsf{K} \in \mathscr{C}_d$, we have

$$\begin{aligned}\mathrm{Vol}_d(\lambda_1(\mathsf{C} \cup \mathsf{K}) + \lambda_2\mathsf{D}_2 + \cdots \lambda_d\mathsf{D}_d) + \mathrm{Vol}_d(\lambda_1(\mathsf{C} \cap \mathsf{K}) + \lambda_2 D_2 + \cdots + \lambda_d\mathsf{D}_d)& \\ = \mathrm{Vol}_d(\lambda_1\mathsf{C} + \lambda_2\mathsf{D}_2 + \cdots + \lambda_d\mathsf{D}_d) + \mathrm{Vol}_d(\lambda_1\mathsf{K} + \lambda_2\mathsf{D}_2 + \cdots + \lambda_d\mathsf{D}_d)&.\end{aligned}$$

We can now apply Theorem 17.2.1 to each summand. Conclude by matching coefficients of the resulting polynomials. □

**Proposition 17.4.6** (Continuity)**.** *The mixed volume is Hausdorff continuous in each coordinate. That is, for all* $\mathsf{D}_2, \dots, \mathsf{D}_d \in \mathscr{C}_d$ *and sequences* $\{\mathsf{C}_i\}_{i\in\mathbb{N}} \subset \mathscr{C}_d$ *with* $\mathsf{C}_i \to \mathsf{C}$*, we have*

$$V(\mathsf{C}_i, \mathsf{D}_2, \dots, \mathsf{D}_d) \to V(\mathsf{C}, \mathsf{D}_1, \dots, \mathsf{D}_d).$$

*Proof.* This follows from the fact that $\mathrm{Vol}_d$ is continuous and that pointwise convergence of a series of polynomials of constant degree implies the convergence of the polynomial coefficients. □

## 17.5 Monotonicity of Mixed Volumes

The mixed volumes are increasing with respect to set inclusion in each coordinate. This fact is rather more difficult to prove as compared with the other properties. In this section, we will give the geometric argument that supports monotonicity.

**Theorem 17.5.1** (Monotonicity)**.** *For* $\mathsf{C}_1, \dots, \mathsf{C}_d, \mathsf{K}_1, \dots, \mathsf{K}_d \in \mathscr{C}_d$*, we have*

$$0 \le V(\mathsf{C}_1, \dots, \mathsf{C}_d) \le V(\mathsf{K}_1, \dots, \mathsf{K}_d). \tag{17.5.1}$$

*Proof of Theorem 17.5.1.* The first inequality in Theorem 17.5.1 follows from the second. Indeed, for $\boldsymbol{x}_i \in \mathsf{C}_i$, we have

$$V(\mathsf{C}_1, \dots, \mathsf{C}_d) \ge V(\{\boldsymbol{x}_1\}, \dots, \{\boldsymbol{x}_d\}) = 0.$$

The last identity is an immediate consequence of the definition of mixed volumes.

To prove the second inequality, we need the following lemma.

**Lemma 17.5.2** (Facet representation of mixed voumes)**.** *Let* $\mathsf{C} \in \mathscr{C}_d$*, and let* $\mathsf{P}_2, \dots, \mathsf{P}_d \in \mathscr{P}_d$ *be polytopes. Introduce the finite set* $\mathscr{U}$ *that contains the unit outer normals of the facets of* $\lambda_2\mathsf{P}_2 + \dots + \lambda_d\mathsf{P}_d$ *for all* $\lambda_i \ge 0$*. Then*

$$V(\mathsf{C}, \mathsf{P}_2, \dots, \mathsf{P}_d) = \frac{1}{d} \sum\nolimits_{\boldsymbol{u}\in\mathscr{U}} h_{\mathsf{C}}(\boldsymbol{u}) \cdot V_{[d-1]}(\mathsf{F}_{\mathsf{P}_1}(\boldsymbol{u}), \dots, \mathsf{F}_{\mathsf{P}_d}(\boldsymbol{u})).$$

*As usual,* $\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})$ *denotes the face of* $\mathsf{P}$ *exposed in the direction* $\boldsymbol{u}$*. We have written* $V_{[d-1]}$ *for the mixed volume in dimension* $d-1$*, where we translate each set* $\mathsf{F}_{\mathsf{P}_i}(\boldsymbol{u})$ *into the hyperplane* $\boldsymbol{u}^\perp$ *containing the origin.*

With this result at hand, we can complete the proof that mixed volumes are monotone. We establish the lemma in the next section.

By continuity of the mixed volume, it is enough to consider the case where $\{\mathsf{C}_i\}_{2\le i\le d}$ and $\{\mathsf{K}_i\}_{2\le i\le d}$ consist of polytopes. By symmetry, it is enough to check monotonicity in the first argument. Therefore, we want to show

$$V(\mathsf{C}, \mathsf{D}_2 \dots, \mathsf{D}_d) \le V(\mathsf{K}, \mathsf{D}_2, \dots, \mathsf{D}_d), \tag{17.5.2}$$

for all polytopes $\mathsf{C}, \mathsf{K}, \mathsf{D}_2, \dots, \mathsf{D}_d \in \mathscr{P}$ with $\mathsf{C} \subset \mathsf{K}$.

We prove (17.5.2) by induction over the dimension, noting that the result is trivial for $d = 1$. Assume that (17.5.2) holds in dimension $d-1$, which implies that $(d-1)$-dimensional mixed volumes are nonnegative. Then we can apply Lemma 17.5.2 to obtain a representation

$$V(\mathsf{C}, \mathsf{D}_2 \dots, \mathsf{D}_d) = \frac{1}{d} \sum\nolimits_{\boldsymbol{u}\in\mathscr{U}} h_{\mathsf{C}}(\boldsymbol{u}) \cdot V_{[d-1]}(\mathsf{F}_{\mathsf{D}_1}(\boldsymbol{u}), \dots, \mathsf{F}_{\mathsf{D}_d}(\boldsymbol{u})).$$

Since $\mathsf{C} \subset \mathsf{K}$, the support functions satisfy $h_{\mathsf{C}} \le h_{\mathsf{K}}$ pointwise. By the inductive hypothesis, $V_{[d-1]}$ is nonnegative. The inequality (17.5.2) follows. □

### 17.5.1 Facet Representation of Generalized Surface Area

We conclude with a formula for the $\mathsf{C}$-surface area of a polytope $\mathsf{P}$. This result leads directly to the mixed volume representation in Lemma 17.5.2.

**Lemma 17.5.3** (Facet representation of generalized surface area). *Let $\mathsf{C} \in \mathscr{C}_d$, and let $\mathsf{P} \in \mathscr{P}_d$ be a polytope. Define $\mathscr{U}$ to be the set of outer normals of facets of $\mathsf{P}$. Then*

$$V(\mathsf{C}, \mathsf{P}, \dots, \mathsf{P}) = \frac{1}{d} \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{C}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})). \tag{17.5.3}$$

Before we prove this result, let us explain how it implies Lemma 17.5.2.

*Proof of Lemma 17.5.2 from Lemma 17.5.3.* Let $\mathsf{P} = \sum_{i=2}^{d} \lambda_i \mathsf{P}_i$ in Lemma 17.5.3. Expand the volume polynomial and extract the mixed derivative. □

Finally, let us establish the facet representation of the generalized surface area of a polytope.

*Proof of Lemma 17.5.3.* First, for a $(d-2)$-dimensional face $\mathsf{G}$ of the polytope $\mathsf{P}$, we have $\mathrm{Vol}_d(\mathsf{G} + \varepsilon \mathsf{B}_d) = \mathscr{O}(\varepsilon^2)$. because of Steiner's formula and the fact $V_d(\mathsf{G}) = V_{d-1}(\mathsf{G}) = 0$.

The next step is analogous to the computation of Minkowski surface area, but we use Minkowski's mixed volume theorem (17.3.2) instead of Steiner's formula. Compute

$$\begin{aligned}
&\frac{1}{d} \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\varepsilon \mathsf{C} + \mathsf{P}) - \mathrm{Vol}_d(\mathsf{P})}{\varepsilon} \\
&\quad = \frac{1}{d} \lim_{\varepsilon \downarrow 0} \frac{\sum_{j=0}^{d} \varepsilon^j \binom{d}{j} V(\mathsf{C}, \dots, \mathsf{C}, \mathsf{P}, \dots, \mathsf{P}) - \varepsilon^0 V(\mathsf{P}, \dots, \mathsf{P})}{\varepsilon} \\
&\quad = \frac{1}{d} \lim_{\varepsilon \downarrow 0} \frac{\sum_{j=1}^{d} \varepsilon^j \binom{d}{j} V(\mathsf{C}, \dots, \mathsf{C}, \mathsf{P}, \dots, \mathsf{P})}{\varepsilon} = V(\mathsf{C}, \mathsf{P}, \dots, \mathsf{P}).
\end{aligned}$$

This gives a geometric expression for the mixed volume we are studying.

Let us establish the formula (17.5.3) in case $\mathbf{0} \in \mathsf{C}$. By a geometric argument, we have

$$\begin{aligned}
\frac{1}{\varepsilon}[\mathrm{Vol}_d(\varepsilon \mathsf{C} + \mathsf{P}) - \mathrm{Vol}_d(\mathsf{P})] &= \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{C}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})) + \frac{1}{\varepsilon} \mathscr{O}(\varepsilon^2) \\
&\longrightarrow \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{C}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})) \quad \text{as } \varepsilon \downarrow 0.
\end{aligned} \tag{17.5.4}$$

Indeed, the only significant contributions to $\mathrm{Vol}_d(\varepsilon \mathsf{C} + \mathsf{P}) - \mathrm{Vol}_d(\mathsf{P})$ come from facets of $\mathsf{P}$. The facet with normal $\boldsymbol{u}$ is extended into a prism by a segment of length $\varepsilon h_{\mathsf{C}}(\boldsymbol{u})$ in a direction chosen from $\mathsf{F}_{\mathsf{C}}(\boldsymbol{u})$; the sum in (17.5.4) totals the volumes of these prisms. The other faces of $\mathsf{P}$ contribute a negligible amount by the first observation in the proof. See Figure 17.4 for an illustration.

Finally, let us extend (17.5.3) to the case where $\mathbf{0} \notin \mathsf{C}$. For a point $\boldsymbol{x} \in \mathsf{C}$, we can apply (17.5.3) to $\mathsf{C} - \boldsymbol{x}$. We obtain

$$\begin{aligned}
V(\mathsf{C}, \mathsf{P}, \dots, \mathsf{P}) &= V(\mathsf{C} - \boldsymbol{x}, \mathsf{P}, \dots, \mathsf{P}) \\
&= \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{C} - \boldsymbol{x}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})) \\
&= \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{C}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})) - \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} \langle \boldsymbol{u}, \boldsymbol{x} \rangle \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})) \\
&= \sum\nolimits_{\boldsymbol{u} \in \mathscr{U}} h_{\mathsf{C}}(\boldsymbol{u}) \cdot \mathrm{Vol}_{d-1}(\mathsf{F}_{\mathsf{P}}(\boldsymbol{u})).
\end{aligned}$$

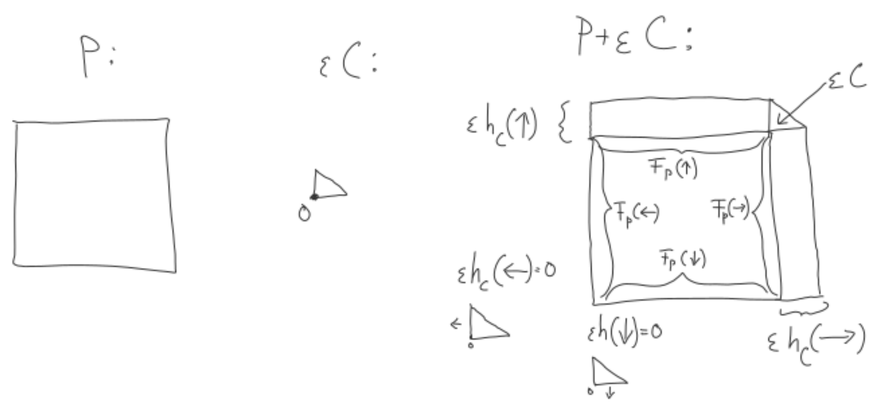


**Figure 17.4** (Facet representation of generalized surface area). Illustration of Equation (17.5.4). The dominant contribution to the volume of $\mathsf{P} + \varepsilon\mathsf{C}$ comes from $\mathsf{P}$ and from the facets of $\mathsf{P}$. The facet with normal $\boldsymbol{u}$ is extended to a prism by a segment of length $\varepsilon h_{\mathsf{C}}(\boldsymbol{u})$ pointing in a direction from $\mathsf{F}_{\mathsf{C}}(\boldsymbol{u})$.

We showed that the second sum in the penultimate line is zero on the homework. □

# Lecture 18: Strongly Isomorphic Polytopes

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*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*4 December 2018*

## 18.1 Agenda for Lecture 18

This week we will prove the Alexandrov–Fenchel Inequality (AFI), a deep fact about mixed volumes. The lecture today contains some geometric preparations that date back to the work of Alexandrov. It describes a method for approximating convex bodies by polytopes with extra properties. This construction will reduce the proof of the AFI to a linear algebra problem.

1. Strongly isomorphic polytopes
2. Simple polytopes
3. Approximation by simple, strongly isomorphic polytopes
4. Support vectors and facet structure
5. Linear extension of mixed volumes

## 18.2 Strongly Isomorphic Polytopes

As part of the proof of Minkowski's theorem on mixed volumes, we established an interested fact about linear families of polytopes. For any polytopes $\mathsf{P}_1, \mathsf{P}_2, \dots, \mathsf{P}_n \subset \mathbb{R}^d$, there is a fixed, finite set $\mathscr{U} = \{u_1, \dots, u_N\} \subset \mathbb{S}^{d-1}$ that contains all outward unit normals of the facets of polytopes of the form

$$\lambda_1 \mathsf{P}_1 + \cdots + \lambda_n \mathsf{P}_n, \quad \text{for all } \lambda_i \geq 0.$$

In other words, forming nonnegative Minkowski combinations of polytopes only generates a finite number of facet directions.

This fact is surprising at first sight, but it is a general property of polytopes. To begin this lecture, we will introduce families of polytopes that exhibit the same behavior in an even purer form.

**Definition 18.2.1** (Strongly isomorphic polytopes)**.** Two polytopes $\mathsf{P}_1, \mathsf{P}_2 \subset \mathbb{R}^d$ are *strongly isomorphic* (SI) if

$$\dim \mathsf{F}_{\mathsf{P}_1}(\boldsymbol{u}) = \dim \mathsf{F}_{\mathsf{P}_2}(\boldsymbol{u}) \quad \text{for all } \boldsymbol{u} \in \mathbb{S}^{d-1}.$$

We can extend this definition to larger families of polytopes, including infinite families. See Figure 18.1 for an illustration.

The basic facts about strongly isomorphic polytopes are not especially hard to establish, but the arguments are not very illuminating. Instead, we summarize some of their basic properties without proof.

**Fact 18.2.2** (Strongly isomorphic polytopes)**.** *Strongly isomorphic polytopes have the following properties.*

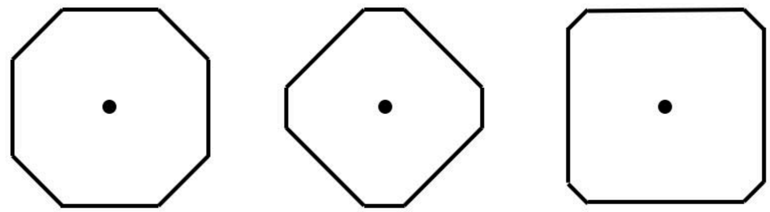


**Figure 18.1** (Strongly isomorphic polytopes). Strongly isomorphic polytopes have the property that the face exposed in a given direction has the same dimension in each polytope. This diagram exhibits a family of three strongly isomorphic polytopes in $\mathbb{R}^2$.

1. *Normal cone characterization. Two polytopes* $\mathsf{P}_1, \mathsf{P}_2 \subset \mathbb{R}^d$ *are SI if and only if the normal cones at vertices are the same. That is, the sets*

$$\{\mathsf{N}_{\mathsf{P}_i}(\boldsymbol{v}) : \boldsymbol{v} \text{ is a vertex of } \mathsf{P}_i\}$$

*are identical for* $i = 1, 2$.

2. *Linear families. If* $\mathsf{P}_1, \mathsf{P}_2 \subset \mathbb{R}^d$ *are SI polytopes, then* $\{\lambda_1 \mathsf{P}_1 + \lambda_2 \mathsf{P}_2 : \lambda_1, \lambda_2 \geq 0\}$ *is an SI family of polytopes.*
3. *Heritability. Let* $\mathsf{P}_1, \mathsf{P}_2 \subset \mathbb{R}^d$ *be SI polytopes. For each fixed* $\boldsymbol{u} \in \mathbb{S}^{d-1}$*, the faces* $\mathsf{F}_{\mathsf{P}_1}(\boldsymbol{u})$ *and* $\mathsf{F}_{\mathsf{P}_2}(\boldsymbol{u})$ *are SI polytopes.*

Recall that a polytope is completely determined by the heights of its facets and the normal directions of its facets. For a family of SI polytopes $\mathscr{S}$, the normal directions of the facets fall in the same finite set. Let

$$\mathscr{U} := \{\boldsymbol{u} \in \mathbb{S}^{d-1} : \mathsf{F}_{\mathsf{P}}(\boldsymbol{u}) \text{ is a facet of } \mathsf{P} \text{ for } \mathsf{P} \in \mathscr{S}\}.$$

Each polytope $\mathsf{P} \in \mathscr{S}$ is determined by the values of its support function $h(\cdot; \mathsf{P})$ on the set $\mathscr{U}$:

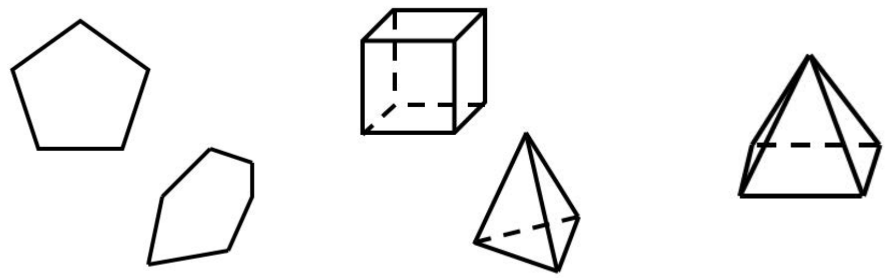
$$\{h(\boldsymbol{u}; \mathsf{P}) : \boldsymbol{u} \in \mathscr{U}\}$$

This observation leads to a definition that will play an important role in the treatment of the Alexandrov–Fenchel inequality.

**Definition 18.2.3** (Support vector)**.** Let $\mathscr{S}$ be a family of SI polytopes in $\mathbb{R}^d$, and let $\mathscr{U} = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_N\}$ be the common set of facet directions. The *support vector* of a polytope $\mathsf{P} \in \mathscr{S}$ is

$$\boldsymbol{h}(\mathsf{P}) := (h_i(\mathsf{P}))_{i=1}^N := (h(\boldsymbol{u}_i; \mathsf{P}))_{i=1}^N \in \mathbb{R}^N.$$

The support vector determines the polytope $\mathsf{P}$.



**(a)** Simple **(b)** Simple **(c)** Not simple

**Figure 18.2** (Simple Polytopes). Each vertex of a simple polytope in $\mathbb{R}^d$ is contained in exactly $d$ facets. (a) In $\mathbb{R}^2$, all convex polygons are simple. (b) The cube and the tetrahedron in $\mathbb{R}^3$ are simple. (c) The square pyramid in $\mathbb{R}^3$ is not simple, because its apex belongs to four facets.

## 18.3 Simple Polytopes

Next, we consider a special class of polytopes that are stable under perturbations. These polytopes also play a basic role in combinatorics and in optimization.

**Definition 18.3.1** (Simple Polytope)**.** A polyhedron $\mathsf{P} \in \mathbb{R}^d$ is *simple* if every vertex in contained in exactly $d$ facets. A *simple polytope* is a simple polyhedron that is also bounded. See Figure 18.2 for an illustration.

Most polytopes are simple, and they remain so after perturbation.

**Fact 18.3.2** (Simple polytopes)**.** *Simple polytopes have the following properties.*

1. *Genericity. Most polytopes are simple. In particular, for any matrix $\boldsymbol{A}$ with full row-rank, the set*

$$\mathsf{P} = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$$

*is a simple polyhedron for almost every right-hand side $\boldsymbol{b}$.*

2. *Perturbations. If $\mathsf{P}$ is a simple polytope, all small perturbations of the facets parallel to their outer normals result in a simple polytope $\mathsf{P}'$ with the same set of outer normals. Moreover, the strong isomorphism class of a simple polytope is preserved by perturbation.*

*Proof sketch.* In the construction of the polyhedron, each inequality that holds with equality determines a facet of $\mathsf{P}$. A vertex of $\mathsf{P}$ is a point where $d$ or more inequalities hold with equality. But, for a generic choice of $\boldsymbol{b}$, it is not possible for more than $d$ of the inequalities to hold with equality at a given point. A similar observation explains why simple polytopes are stable under perturbation. The result on perturbations of strongly isomorphic polytopes involves reasoning about normal cones. □

## 18.4 Approximation by Simple, Strongly Isomorphic Polytopes

Strongly isomorphic polytopes are completely determined by their support vectors. Simple polytopes are stable under perturbation. We wish to work with polytopes that enjoy both of these favorable properties. This requirement may seem restrictive. But, in fact, these polytopes are everywhere.

**Theorem 18.4.1** (Approximation by SSI polytopes). *Let $\mathsf{C}_1, \dots, \mathsf{C}_m \subset \mathbb{R}^d$ be convex bodies containing the origin. For all $\varepsilon > 0$, there are simple, strongly isomorphic (SSI) polytopes $\mathsf{P}_1, \dots, \mathsf{P}_m \subset \mathbb{R}^d$ such that*

$$\operatorname{dist}_{\mathrm{H}}(\mathsf{C}_i, \mathsf{P}_i) < \varepsilon \quad \textit{and} \quad \mathbf{0} \in \operatorname{int} \mathsf{P}_i \quad \textit{for each } i = 1, \dots, m.$$

*Proof sketch.* For each $i$, approximate $\mathsf{C}_i$ by a polytope $\mathsf{Q}_i$ with $\mathbf{0} \in \operatorname{int} \mathsf{Q}_i$. Let $\mathsf{P} = \mathsf{Q}_1 + \dots + \mathsf{Q}_m$. There is a perturbation $\mathsf{P}'$ of $\mathsf{P}$ that is a simple polytope that has a facet parallel to every facet of $\mathsf{P}$. For very small $\alpha > 0$, define $\mathsf{P}_i = \mathsf{Q}_i + \alpha \mathsf{P}'$. □

Here is the key outcome of this result. Given a family of $m$ convex bodies, we can approximate it with a family $(\mathsf{P}_1, \dots, \mathsf{P}_d)$ of $m$ SSI polytopes with $N$ facet-normal directions. Consider the set of polytopes

$$\mathscr{S} = \{\mathsf{P} : \mathsf{P} \text{ shares the isomorphism class of } \mathsf{P}_i\}.$$

Note that $\mathscr{S}$ contains all small facet-normal perturbations of each simple polytope $\mathsf{P} \in \mathscr{S}$. One particular consequence is that

$$\operatorname{lin}\{\boldsymbol{h}(\mathsf{P}) : \mathsf{P} \in \mathscr{S}\} = \mathbb{R}^N.$$

This result will allow us to construct a linear extension of the mixed volumes later in this lecture.

## 18.5 Support Vectors and Facet Structure

As we have seen, the support vector of a polytope (in an SI family) determines the polytope completely. Our next goal is to develop more concrete realizations of this principle. In particular, we will explain how the support vector of the polytope determines the structure of its facets.

### 18.5.1 Neighboring Facets

Let $\mathscr{S}$ be an SI family with common unit normal directions $\mathscr{U} = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_N\}$. Fix a polytope $\mathsf{P} \in \mathsf{S}$. Define the facet of $\mathsf{P}$ exposed in the direction $\boldsymbol{u}_i$:

$$\mathsf{F}_i := \mathsf{F}_i(\mathsf{P}) := \mathsf{F}_{\mathsf{P}}(\boldsymbol{u}_i) \quad \text{for } i = 1, \dots, N.$$

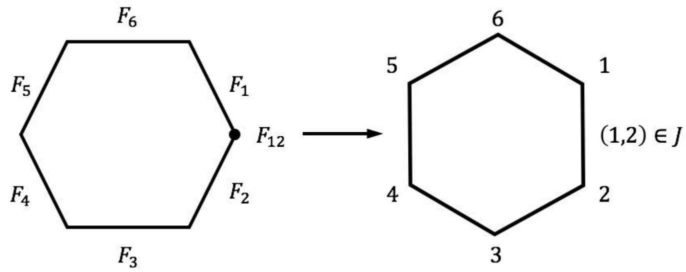
For each pair of facets, let

$$\mathsf{F}_{ij} := \mathsf{F}_{ij}(\mathsf{P}) := \mathsf{F}_i(\mathsf{P}) \cap \mathsf{F}_j(\mathsf{P}).$$

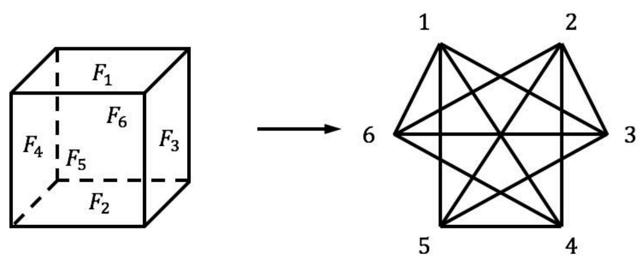
The set $\mathsf{F}_{ij}$ is either empty or has dimension $d-2$. In the latter case, $\mathsf{F}_{ij}$ is a ridge of the polytope $\mathsf{P}$. Introduce the set

$$J := \{(i,j) \subset \mathbb{R}^{N \times N} : \dim \mathsf{F}_{ij} = d - 2\}.$$

The set $J$ contains the information about which pairs of facets are adjacent or neighboring. The following observation is immediate.




**(a)** (A convex polygon). In $\mathbb{R}^2$, the hexagon has six facets, each with two neighbors. The set $J$ of neighboring facets is a cycle on the six facets.




**(b)** (A cube). In $\mathbb{R}^3$, the cube has six facets, each with four neighbors. The set $J$ of neighboring facets is a clique, minus the three edges $((1,2),(3,4),(5,6))$ associated with opposite pairs of faces.

**Figure 18.3** (Facets and neighboring facets). This diagram contains examples of two convex polytopes, their facets, and the graph determined by neighboring facets.

**Fact 18.5.1** (Neighboring facets). *The set $J$ determines the edges of a connected, undirected graph on the facets $\{1,\dots,N\}$.*

See Figure 18.3 for an illustration of these concepts.

### 18.5.2 The Support Function of a Facet

Let $\mathsf{F}_i$ be a fixed facet of the polytope $\mathsf{P}$, and let $\mathsf{H}_i$ be the (unique) hyperplane that contains the facet $\mathsf{F}_i$. Our goal is to compute the support function of $\mathsf{F}_i$ with respect to the plane $\mathsf{H}_i$ in terms of the support vector $\boldsymbol{h} = \boldsymbol{h}(\mathsf{P})$. The reason for this particular computation will become clear when we use these results to develop formulas for mixed volumes.

Each facet of $\mathsf{F}_i$ is a ridge $\mathsf{F}_{ij}$ of the polytope $\mathsf{P}$. Therefore, each facet of $\mathsf{F}_i$ is determined by the intersection of $\mathsf{F}_i$ with $\mathsf{F}_j$ where $(i,j) \in J$. In other words, each facet of $\mathsf{F}_i$ is exposed by the direction $\boldsymbol{u}_j$ where $(i,j) \in J$. We need to project $\boldsymbol{u}_j$ onto the plane $\mathsf{H}_i$ to obtain the outer normal of the facet $\mathsf{F}_{ij}$, and this requires some trigonometry.

We wish to compute the level of the facet $\mathsf{F}_{ij}$ of $\mathsf{F}_i$. To that end, define the angle between the facet normals:

$$\theta_{ij} := \measuredangle(\boldsymbol{u}_i, \boldsymbol{u}_j) \in (0, \pi) \quad \text{for } (i,j) \in J.$$

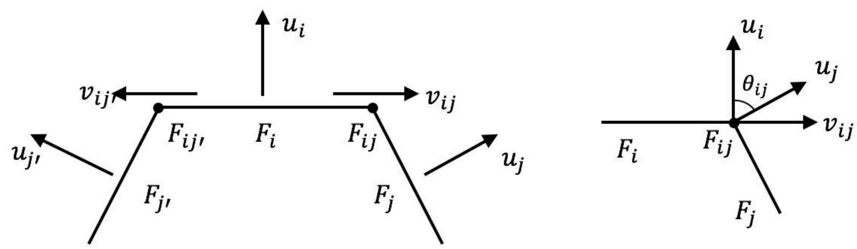



**Figure 18.4** (Facets of facets). [left] The set $\mathsf{F}_{ij}$ is a facet of $\mathsf{F}_i$, exposed in the direction $\boldsymbol{u}_j$. The vector $\boldsymbol{v}_{ij}$ is the unit outer normal of the facet $\mathsf{F}_{ij}$, considered as a subset of $\operatorname{aff} \mathsf{F}_i$. This diagram also illustrates a second facet $\mathsf{F}_{ij'}$ of $\mathsf{F}_i$. [right] Computation of the support function of $\mathsf{F}_{ij}$ in the direction $\boldsymbol{v}_{ij}$.

Define the vector $\boldsymbol{v}_{ij} \perp \boldsymbol{u}_i$ to be the unit outer normal to $\mathsf{F}_{ij}$, regarded as a facet of $\mathsf{F}_i$. In other words, we need the value of the support function of $\mathsf{F}_i$ in the direction $\boldsymbol{v}_{ij}$:

$$h_{ij} := h_{ij}(\mathsf{P}) := \max\{\langle \boldsymbol{v}_{ij}, \boldsymbol{x}\rangle : \boldsymbol{x} \in \mathsf{F}_i\} = \langle \boldsymbol{v}_{ij}, \boldsymbol{x}_j\rangle \quad \text{for any } \boldsymbol{x}_j \in \mathsf{F}_{ij}.$$

See Figure 18.4 for an illustration.

By plane trigonometry,

$$\boldsymbol{u}_j = \boldsymbol{u}_i \cos\theta_{ij} + \boldsymbol{v}_{ij} \sin\theta_{ij}.$$

Take the inner product of this expression with any vector $\boldsymbol{x}_j \in \mathsf{F}_{ij} = \mathsf{F}_i \cap \mathsf{F}_j$. Then

$$\langle \boldsymbol{u}_j, \boldsymbol{x}_j\rangle = \langle \boldsymbol{u}_i, \boldsymbol{x}_j\rangle \cos\theta_{ij} + \langle \boldsymbol{v}_{ij}, \boldsymbol{x}_j\rangle \sin\theta_{ij}.$$

Identify the support levels of the facets $\mathsf{F}_j, \mathsf{F}_i, \mathsf{F}_{ij}$ to see that

$$h_j = h_i \cos\theta_{ij} + h_{ij} \sin\theta_{ij}.$$

Solve this expression for $h_{ij}$ to reach

$$h_{ij} = h_j \csc\theta_{ij} - h_i \cot\theta_{ij} \quad \text{where } (i,j) \in J. \tag{18.5.1}$$

In this relation, $h_i = h_i(\mathsf{P})$.

To reiterate, we have computed the support vector $\boldsymbol{h}_{i:}$ of the facet $\mathsf{F}_i$ in terms of the support vector $\boldsymbol{h}$ of $\mathsf{P}$. It is important to note that $\boldsymbol{h}_{i:}$ is a linear function of $\boldsymbol{h}$.

## 18.6 Linear Extension of Mixed Volumes

Next, we will show how to express mixed volumes in terms of support vectors and to extend this formula linearly. We have the following process:

$$V(\mathsf{C}_1, \dots, \mathsf{C}_d) \underset{\text{SSI}}{\overset{\text{approx}}{\rightleftharpoons}} V(\mathsf{P}_1, \dots, \mathsf{P}_d)$$
$$\underset{\text{restrict to facets}}{\overset{\text{supp. vector of facets}}{\rightleftharpoons}} V(\boldsymbol{h}(\mathsf{P}_1), \dots, \boldsymbol{h}(\mathsf{P}_d)).$$

In other words, we can approximate a family of convex bodies by a family of SSI polytopes. Each of these polytopes is determined completely by its support vector, i.e., the height of its facets. As such, we may as well compute mixed volumes of SSI bodies directly in terms of their support functions.

### 18.6.1 The Two-Dimensional Case

To warm up, let us work out what happens in $\mathbb{R}^2$. Let $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^2$ be a pair of SSI polygons with $N$ facets. The set $J$ of neighboring facets is always a cycle graph:

$$J = \{(i, (i+1) \bmod N) : i = 1, \dots, N\}.$$

In other words, the $i$th facet is contingent on the facets $i+1$ and $i-1$. Introduce the support vectors $\boldsymbol{h}(\mathsf{C}), \boldsymbol{h}(\mathsf{K}) \in \mathbb{R}^N$ of the sets, along with the support vectors $\boldsymbol{h}_{i:}(\mathsf{K})$ of the facets of $\mathsf{K}$.

Using the facet representation of the mixed volume, we calculate that

$$\begin{aligned}
V(\mathsf{C}, \mathsf{K}) &= \frac{1}{2} \sum\nolimits_{i=1}^{N} h_i(\mathsf{C}) \cdot \mathrm{Vol}_1(\mathsf{F}_i(\mathsf{K})) \\
&= \frac{1}{2} \sum\nolimits_{i=1}^{N} h_i(\mathsf{C}) (h_{i,i+1}(\mathsf{K}) + h_{i,i-1}(\mathsf{K})) \\
&= \frac{1}{2} \sum\nolimits_{i=1}^{N} h_i(\mathsf{C}) \big[ h_{i+1}(\mathsf{K}) \csc\theta_{i,i+1} + h_{i-1}(\mathsf{K}) \csc\theta_{i,i-1} \\
&\qquad\qquad - h_i(\mathsf{K})(\cot\theta_{i,i+1} + \cot\theta_{i,i-1}) \big] \\
&= \frac{1}{2} \sum\nolimits_{(i,j) \in J} h_i(\mathsf{C}) h_j(\mathsf{K}) \csc\theta_{ij} - \frac{1}{2} \sum\nolimits_{i=1}^{N} h_i(\mathsf{C}) h_i(\mathsf{K}) (\cot\theta_{i,i+1} + \cot\theta_{i,i-1}) \\
&= \frac{1}{2} \boldsymbol{h}(\mathsf{C})^* \boldsymbol{M} \boldsymbol{h}(\mathsf{K}) - \frac{1}{2} \boldsymbol{h}(\mathsf{C})^* \boldsymbol{D} \boldsymbol{h}(\mathsf{K}) \\
&= \frac{1}{2} \boldsymbol{h}(\mathsf{C})^* (\boldsymbol{M} - \boldsymbol{D}) \boldsymbol{h}(\mathsf{K}).
\end{aligned}$$

In the second line, we have used the fact that the length of the one-dimensional set $\mathsf{F}_i(\mathsf{K})$ is the sum of its two support values $h_{i,i+1}(\mathsf{K})$ and $h_{i,i-1}(\mathsf{K})$. In the third line, we introduce the formula (18.5.1). Last, rearrange this expression, and write it in matrix form.

Now, observe that the matrix $\boldsymbol{M} \in \mathbb{R}^{N \times N}$ is nonnegative and irreducible:

$$(\boldsymbol{M})_{ij} = \begin{cases} \csc\theta_{ij}, & \text{for } (i,j) \in J, \\ 0, & \text{otherwise.} \end{cases}$$

The irreducibility holds because $J$ is the adjacency matrix of a connected, undirected graph and $\csc\theta_{ij} > 0$. Meanwhile, the matrix $\boldsymbol{D} \in \mathbb{R}^{N \times N}$ is diagonal:

$$(\boldsymbol{D})_{ii} = \cot\theta_{i,i+1} + \cot\theta_{i,i-1} \quad \text{for } i = 1, \dots, N.$$

We have expressed the mixed volume as a bilinear form with very concrete properties.

Finally, recall that the support vectors $\boldsymbol{h}(\mathsf{P})$ of polytopes in the same isomorphism class of $\mathsf{C}$ and $\mathsf{K}$ span $\mathbb{R}^N$. Therefore, we can extend this computation to all of $\mathbb{R}^N$ using linearity.

$$V(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^* (\boldsymbol{M} - \boldsymbol{D}) \boldsymbol{y} \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N.$$

This expression gives concrete meaning to the mixed volume of an arbitrary pair of support vectors.

### 18.6.2 The General Case

We can do the same thing in higher dimensions, but the formulas become more tangled. Let $\mathsf{P}_1, \dots, \mathsf{P}_d \subset \mathbb{R}^d$ be an SSI family of polytopes with support vectors $\boldsymbol{h}(\mathsf{P}_i) \in \mathbb{R}^N$. Let $J$ be the facet graph of this isomorphism class. Using the facet representation of mixed volumes (twice!),

$$\begin{aligned} V(\mathsf{P}_1, \dots, \mathsf{P}_d) &= \frac{1}{d} \sum\nolimits_{i=1}^N h_i(\mathsf{P}_1) \cdot V(\mathsf{F}_i(\mathsf{P}_2), \dots, \mathsf{F}_i(\mathsf{P}_d)) \\ &= \frac{1}{d(d-1)} \sum\nolimits_{i=1}^N h_i(\mathsf{P}_1) \sum\nolimits_{(i,j)\in J} h_{ij}(\mathsf{P}_2) \cdot V(\mathsf{F}_{ij}(\mathsf{P}_3), \dots, \mathsf{F}_{ij}(\mathsf{P}_d)) \\ &= \dots \\ &= \frac{1}{d} \boldsymbol{h}(\mathsf{P}_1)^* (\boldsymbol{M} - \boldsymbol{D}) \boldsymbol{h}(\mathsf{P}_2). \end{aligned}$$

Note that we have used the computation of the support of $\mathsf{F}_i$ in the plane $\mathsf{H}_i$ to apply the facet representation of the mixed volume the second time. The rest of the details are similar.

In this case, the matrix $\boldsymbol{M}$ remains nonnegative and irreducible:

$$(\boldsymbol{M})_{ij} = \begin{cases} (d-1)^{-1} \csc \theta_{ij} \cdot V(\mathsf{F}_{ij}(\mathsf{P}_3), \dots, \mathsf{F}_{ij}(\mathsf{P}_d)), & \text{for } (i,j) \in J, \\ 0, & \text{otherwise.} \end{cases}$$

The entries $(\boldsymbol{M})_{ij} > 0$ for $(i,j) \in J$ because the $(d-2)$-dimensional mixed volume $V(\mathsf{F}_{ij}(\mathsf{P}_3), \dots, \mathsf{F}_{ij}(\mathsf{P}_d))$ of the $d-2$ ridges is always positive. The matrix $\boldsymbol{D}$ is diagonal:

$$\begin{aligned} (\boldsymbol{D})_{ii} = \frac{1}{d-1} \sum\nolimits_{j:(i,j)\in J} \big[ &\cot \theta_{ij} \cdot V(\mathsf{F}_{ij}(\mathsf{P}_3), \dots, \mathsf{F}_{ij}(\mathsf{P}_d)) \\ &+ \cot \theta_{ji} \cdot V(\mathsf{F}_{ji}(\mathsf{P}_3), \dots, \mathsf{F}_{ji}(\mathsf{P}_d)) \big]. \end{aligned}$$

As before, we can extend the mixed volumes linearly to all vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$:

$$V(\boldsymbol{x}, \boldsymbol{y}, \mathsf{P}_3, \dots, \mathsf{P}_d) = \frac{1}{d} \boldsymbol{x}^* (\boldsymbol{M} - \boldsymbol{D}) \boldsymbol{y}.$$

These detailed formulas and the linear extensions will play a key role in the proof of the AFI in the next lecture.

# Lecture 19: The Alexandrov–Fenchel Inequality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*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*6 December 2018*


## 19.1 Agenda for Lecture 19

In this course we devoted a lot of time to discuss different concepts of volumes and inequalities between them. These have remarkable geometric consequences. For example, the isoperimetric inequality gives an extremal property for the Euclidean unit ball that implies both the Sobolev inequality (with optimal constants) and concentration of measure phenomena for Lipschitz functions on the sphere. Today, we will discuss the mother of all volume inequalities.

1. The Alexandrov–Fenchel inequality (AFI)
2. Consequences
3. Setup for proof
4. Proof of the AFI

## 19.2 The Alexandrov–Fenchel Inequality

The Alexandrov–Fenchel inequality is a fundamental fact about mixed volumes.

**Theorem 19.2.1** (Alexandrov 1937,1938)**.** *Let* $\mathsf{C}, \mathsf{K}, \mathsf{P}_1, \dots, \mathsf{P}_d \subset \mathbb{R}^{d+2}$ *be convex bodies. Then,*

$$V(\mathsf{C}, \mathsf{K}, \mathsf{P}_1, \dots, \mathsf{P}_d)^2 \geq V(\mathsf{C}, \mathsf{C}, \mathsf{P}_1, \dots, \mathsf{P}_d) \cdot V(\mathsf{K}, \mathsf{K}, \mathsf{P}_1, \dots, \mathsf{P}_d).$$

At first glance, the AFI vaguely resembles the Cauchy–Schwarz inequality, but it goes in the opposite direction! Bounds of this form are called *hyperbolic inequalities* in contrast to *elliptic inequalities*, like Cauchy–Schwarz.

Theorem 19.2.1 is considered to be one of the deepest and most powerful results in all of convex geometry. Until recently, it has also had the reputation of being brutally hard to prove.

The result was first claimed by Fenchel in 1936, but his proof was unsatisfactory. Hadwiger described the argument as "schwer verständlich" (difficult to understand). The first proof was obtained by Alexandrov in 1937 using strongly isomorphic polytopes. Alexandrov developed a second proof in 1938 using differential geometry and inequalities for mixed dicriminants. It is traditional to name the inequality after both Fenchel (for the conjecture) and Alexandrov (for the proof).

Around 1980, Khonvanskii and Tessier independently showed that the AFI is equivalent to the Hodge index theorem from algebraic geometry. This led to other proofs via algebraic methods.

There are further connections between the AFI and mixed discriminants that were until recently opaque, but the relationship has recently been clarified. Mixed discrimants play an important role in discrete geometry.

Today, we will present a proof of the AFI due to Shenfeld & Van Handel from November 2018 [SVH18]. In a sense, their argument "short circuits" Alexandrov's polytope proof. The setup and the machinery are similar in spirit, but Shenfeld & Van Handel discovered that there is a simple device that reduces Alexandrov's difficult computations to a single line.

Shenfeld & Van Handel have also given a short, clean variant of Alexandrov's differential geometry proof that exposes the connection between the AFI and Alexandrov's inequality for mixed discriminants. The differential geometry proof is conceptually simpler than the polytope proof. But it requires substantive results about elliptic partial differential equations that we prefer not to rely on.

## 19.3 Consequences of the Alexandrov–Fenchel Inequality

Using the symmetry of the mixed volumes, we can extract many further inequalities from the AFI by iteration. Before establishing the result, we discuss some of these consequences (without proof).

The AFI implies *Minkowski's first inequality*, which states that

$$V(\mathsf{C},\dots,\mathsf{C},\mathsf{K})^d \geq \mathrm{Vol}_d(\mathsf{C})^{d-1}\cdot \mathrm{Vol}_d(\mathsf{K}).$$

This in turn yields the Brunn–Minkowski theorem for convex bodies and also Urysohn's mean width inequality. In fact, there is a more general lower bound:

$$V(\mathsf{C}_1,\dots,\mathsf{C}_d)^d \geq \mathrm{Vol}_d(\mathsf{C}_1)\cdots \mathrm{Vol}_d(\mathsf{C}_d).$$

That is, the mixed volume is always bigger than the geometric mean of the volumes.

Recall that mixed volumes give rise to intrinsic volumes by combining a convex body with Euclidean balls in different proportions. As a consequence, the AFI yields inequalities for the intrinsic volumes.

In particular, the AFI implies an entire sequence of isoperimetric inequalities:

$$\left(\frac{\mathrm{Vol}_d(\mathsf{C})}{\mathrm{Vol}_d(\mathsf{B}_d)}\right)^{1/d} \leq \left(\frac{V_{d-1}(\mathsf{C})}{V_{d-1}(\mathsf{B}_d)}\right)^{1/(d-1)} \leq \cdots \leq \frac{V_1(\mathsf{C})}{V_1(\mathsf{B}_d)}.$$

The first relation is the isoperimetric inequality. The inequality between the first member and the last is Urysohn's inequality.

The AFI also implies that the intrinsic volume sequence is (ultra-)log-concave. For all $j$ (where it makes sense),

$$V_j(\mathsf{C})^2 \geq \frac{j+1}{j}\cdot V_{j+1}(\mathsf{C})\cdot V_{j-1}(\mathsf{C}).$$

This tells us that intrinsic volumes have a sub-Poissonian distribution. This result is due to Chevet in 1976, and it was rediscovered by McMullen in 1991.

## 19.4 Setup for the Proof

Before we begin the proof, we need to set out some background material.

### 19.4.1 Simple, Strongly Isomorphic Polytopes

We begin with a brief recapitulation of the insights from last lecture. Fix $d \geq 0$. Let $\mathsf{C},\mathsf{K},\mathsf{P}_1,\dots,\mathsf{P}_d \subset \mathbb{R}^{d+2}$ be a family of *simple, strongly isomorphic* (SSI) polytopes. Let $\mathscr{U} = \{\boldsymbol{u}_1,\dots,\boldsymbol{u}_N\} \subset \mathbb{S}^{d+1}$ be the common set of unit outward normals to facets of the polytopes. We define the support vector of a polytope $\mathsf{P}$ in this family to be

$$\boldsymbol{h}(\mathsf{P}) := (h_i(\mathsf{P}))_{i=1}^N = (h(\boldsymbol{u}_i,\mathsf{P}))_{i=1}^N \in \mathbb{R}^N.$$

Consider the family $\mathscr{S}$ of polytopes with the same isomorphism class as $\mathsf{C}, \mathsf{K}, \mathsf{P}_i$. The support vectors of polytopes in $\mathscr{S}$ have some appealing properties.

(i) If $\mathsf{P} \in \mathscr{S}$ is simple, its support vector is still a support vector of a polytope in $\mathscr{S}$ after an arbitrary small perturbation.

(ii) In particular, the support vectors of polytopes in $\mathscr{S}$ span all of $\mathbb{R}^N$.

(iii) The support vectors of polytopes in $\mathscr{S}$ form a convex cone.

Next, we define the facet maps:

$$\mathsf{F}_i(\mathsf{P}) = \text{facet of } \mathsf{P} \text{ exposed in direction } \boldsymbol{u}_i.$$

Recall that (the support vector of) the facet $\mathsf{F}_i(\mathsf{P})$ is a linear function of the support vector $\boldsymbol{h}(\mathsf{P})$. Since the support vectors span all of $\mathbb{R}^N$, we can extend $\mathsf{F}_i(\mathsf{P}) = \mathsf{F}_i(\boldsymbol{h}(\mathsf{P}))$ linearly to obtain a map $\mathsf{F}_i(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^N$. That is, instead of putting support vectors into the facet map, we can input arbitrary vectors. It is not important to assign any particular geometric meaning to this expression.

Now, we come to the most important observation from last lecture. The mixed volumes can be written as

$$\begin{aligned} V(\mathsf{C}, \mathsf{K}, \mathsf{P}_1, \dots, \mathsf{P}_d) &= \frac{1}{d+2} \sum\nolimits_{i=1}^{N} h_i(\mathsf{C}) \cdot V(\mathsf{F}_i(\mathsf{K}), \mathsf{F}_i(\mathsf{P}_1), \dots, \mathsf{F}_i(\mathsf{P}_d)) \\ &= \boldsymbol{h}(\mathsf{C})^*(\boldsymbol{M} - \boldsymbol{D})\boldsymbol{h}(\mathsf{K}). \end{aligned}$$

In this expression, $\boldsymbol{M}$ is a nonnegative, symmetric, irreducible matrix, and $\boldsymbol{D}$ is a diagonal matrix. This bilinear form allows us to extend mixed volumes linearly to all vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$:

$$\begin{aligned} V(\boldsymbol{x}, \boldsymbol{y}, \mathsf{P}_1, \dots, \mathsf{P}_d) &:= \boldsymbol{x}^*(\boldsymbol{M} - \boldsymbol{D})\boldsymbol{y} \\ &=: \frac{1}{d+2} \sum\nolimits_{i=1}^{N} x_i V(\mathsf{F}_i(\boldsymbol{y}), \mathsf{F}_i(\mathsf{P}_1), \dots, \mathsf{F}_i(\mathsf{P}_d)). \end{aligned}$$

We can assign meaning to the first and last expression using the second expression.

### 19.4.2 Irreducible Matrices

The matrix $\boldsymbol{M}$ that appears in the formula for the mixed volumes is nonnegative and irreducible. A great deal is known about the eigenstructure of this type of matrix. We rely on the following well-known result.

**Fact 19.4.1** (Perron). *Suppose that $\boldsymbol{A}$ is a nonnegative, irreducible square matrix. Then the maximum eigenvalue of $\boldsymbol{A}$ is simple, and the associated eigenvector is the unique eigenvector with strictly positive entries.*

This is the linear-algebraic version of the fact that a random walk on a connected graph converges to a stationary distribution that assigns weight to each point. Think of the positive eigenvector as the stationary distribution. We are not going to prove this fact here; it belongs in a lecture about linear algebra or Markov chains.

### 19.4.3 Hyperbolic Inequalities

Let $\boldsymbol{A}$ be a positive-semidefinite matrix. The bilinear form induced by this matrix satisfies a Cauchy–Schwarz inequality:

$$\langle \boldsymbol{x}, \boldsymbol{A}\boldsymbol{y} \rangle^2 \leq \langle \boldsymbol{x}, \boldsymbol{A}\boldsymbol{x} \rangle \langle \boldsymbol{y}, \boldsymbol{A}\boldsymbol{y} \rangle \quad \text{for all } \boldsymbol{x}, \boldsymbol{y}.$$

The results discussed here go into the opposite direction, however. As a consequence, we need to understand circumstances when a bilinear form obeys the opposite inequality. We have the following result.

**Lemma 19.4.2** (Hyperbolic inequalities)**.** *Let $\boldsymbol{A}$ be self-adjoint with respect to a given inner product. The following are equivalent:*

*(i)* *$\boldsymbol{A}$ has at most one eigenvector with a positive eigenvalue.*

*(ii)* *$\langle \boldsymbol{x}, \boldsymbol{A}\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, \boldsymbol{A}\boldsymbol{x} \rangle \langle \boldsymbol{y}, \boldsymbol{A}\boldsymbol{y} \rangle$ for all $\boldsymbol{y}$ such that $\langle \boldsymbol{y}, \boldsymbol{A}\boldsymbol{y} \rangle \geq 0$ and for all $\boldsymbol{x}$.*

The proof is an exercise in linear algebra, and it bears similarities to the proof of Cauchy–Schwarz.

Relations of the kind in Lemma 19.4.2 are called *hyperbolic inequalities*. To see why, consider the following polynomial in $d+1$ variables:

$$q(\boldsymbol{x}) = x_0^2 - x_1^2 - \cdots - x_d^2 = \boldsymbol{x}^* \boldsymbol{J} \boldsymbol{x} \quad \text{where} \quad \boldsymbol{J} = \operatorname{diag}(1, -1, -1, \ldots, -1).$$

This polynomial appears in the abstract theory of space-time. There is a single distinguished direction $x_0$ corresponding to time, while the other $d$ (undistinguished) coordinates correspond to space. Setting this polynomial greater than or equal to zero results in the pair of light cones at the origin. Everything within the future light cone (time is nonnegative) may be affected by a particle emitted at the origin while everything within the past light cone (time is nonpositive) may have affected the origin (future+past causal connections). This behavior arises from the fact that the bilinear form $\boldsymbol{J}$ has the Lorentz spectral signature: one eigenvalue (time) is positive, the remaining ones (space) are all negative.

For the task at hand, the vector $\boldsymbol{y}$ will be a support vector, which will ensure that $\langle \boldsymbol{y}, \boldsymbol{A}\boldsymbol{y} \rangle \geq 0$. But it will be very important that the vector $\boldsymbol{x} \in \mathbb{R}^N$ can be chosen in an arbitrary fashion.

### 19.4.4 A Refined AFI

We are now prepared to write down the version of the AFI that we are going to prove.

**Theorem 19.4.3.** *Let $\mathsf{K}, \mathsf{P}_1, \ldots, \mathsf{P}_d \subset \mathbb{R}^{d+2}$ be a family of simple, strongly isomorphic polytopes. Let $\mathscr{U} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\} \subset \mathbb{S}^{d+1}$ be the common set of facet outer normals. Then, for any $\boldsymbol{x} \in \mathbb{R}^N$,*

$$V(\boldsymbol{x}, \mathsf{K}, \mathsf{P}_1, \ldots, \mathsf{P}_d)^2 \geq V(\boldsymbol{x}, \boldsymbol{x}, \mathsf{P}_1, \ldots, \mathsf{P}_d) \cdot V(\mathsf{K}, \mathsf{K}, \mathsf{P}_1, \ldots, \mathsf{P}_d).$$

In particular, we can take $\boldsymbol{x} = \boldsymbol{h}(\mathsf{C})$, where $\mathsf{C}$ is another simple, strongly isomorphic polytope. This implies AFI, because we can approximate a general family of convex bodies by a family of simple, strongly isomorphic polytopes.

## 19.5 Proof of Theorem 19.4.3

The proof works by induction on the dimension.

### 19.5.1 Base Case

We begin with the base case $d = 0$. Let $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^2$ be simple, strongly isomorphic polytopes. Minkowski's first theorem states

$$V(\mathsf{C}, \mathsf{K})^2 \geq V(\mathsf{C}, \mathsf{C}) \cdot V(\mathsf{K}, \mathsf{K}).$$

We established this result on the homework as a consequence of the Brunn–Minkowski inequality.

For any $\boldsymbol{x} \in \mathbb{R}^N$, the vector $\boldsymbol{x} + \alpha \boldsymbol{h}(\mathsf{K})$ is guaranteed to be a support vector, provided that $\alpha > 0$ is sufficiently large. This is a consequence of the fact that the support vectors form a cone, and the support vector $\boldsymbol{h}(\mathsf{K})$ of the SSI polytope $\mathsf{K}$ remains a support vector after an arbitrary small perturbation.

We can use this fact to infer

$$V(\boldsymbol{x} + \alpha \boldsymbol{h}(\mathsf{K}), \mathsf{K})^2 \geq V(\boldsymbol{x} + \alpha \boldsymbol{h}(\mathsf{K}), \boldsymbol{x} + \alpha \boldsymbol{h}(\mathsf{K})) \cdot V(\mathsf{K}, \mathsf{K}).$$

The base case now follows from the linearity of the mixed volumes, written in terms of the support vectors.

### 19.5.2 The Inductive Step

Let us turn to the induction step. We need to show that, for all $\boldsymbol{x} \in \mathbb{R}^N$,

$$\mathsf{V}(\boldsymbol{x}, \mathsf{P}_1, \dots, \mathsf{P}_d)^2 \geq V(\boldsymbol{x}, \boldsymbol{x}, \mathsf{P}_1, \dots, \mathsf{P}_d) \cdot V(\mathsf{K}, \mathsf{K}, \mathsf{P}_1, \dots, \mathsf{P}_d).$$

We are going to establish this point by exploiting the special structure of the bilinear form associated with mixed volumes.

Translation invariance allows us to assume that $\mathbf{0} \in \operatorname{int}(\mathsf{P}_1)$. This choice ensures that $h_i(\mathsf{P}_1)$ is strictly positive for all $i$. Define a sequence of weights:

$$p_i := \frac{1}{d+2} \frac{V(\mathsf{F}_i(\mathsf{P}_1), \dots, \mathsf{F}_i(\mathsf{P}_d))}{h_i(\mathsf{P})} > 0 \quad \text{for } 1 \leq i \leq N.$$

We use these positive $p_i$ to define a weighted inner product on $\mathbb{R}^N$:

$$\langle \boldsymbol{a},\, \boldsymbol{b} \rangle_{\boldsymbol{p}} := \sum\nolimits_{i=1}^N p_i a_i b_i.$$

Next, construct a matrix $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ via its action on $\boldsymbol{x} \in \mathbb{R}^N$:

$$(\boldsymbol{A}\boldsymbol{x})_i = \frac{1}{d+2} \cdot \frac{1}{p_i} \cdot V_{d+1}(\mathsf{F}_i(\boldsymbol{x}), \mathsf{F}_i(\mathsf{P}_1), \dots, \mathsf{F}_i(\mathsf{P}_d)).$$

The matrix $\boldsymbol{A}$ has several remarkable properties. First, note that

$$\begin{aligned}
\langle \boldsymbol{y},\, \boldsymbol{A}\boldsymbol{x} \rangle_{\boldsymbol{p}} &= \sum\nolimits_{i=1}^N p_i y_i (\boldsymbol{A}\boldsymbol{x})_i \\
&= \frac{1}{d+2} \sum\nolimits_{i=1}^N y_i V(\mathsf{F}_i(\boldsymbol{x}), \mathsf{F}_i(\mathsf{P}_1), \dots, \mathsf{F}_i(\mathsf{P}_d)) \\
&= V(\boldsymbol{y}, \boldsymbol{x}, \mathsf{P}_1, \dots, \mathsf{P}_d) \\
&= V(\boldsymbol{x}, \boldsymbol{y}, \mathsf{P}_1, \dots, \mathsf{P}_d) \\
&= \cdots = \langle \boldsymbol{x},\, \boldsymbol{A}\boldsymbol{y} \rangle_{\boldsymbol{p}}.
\end{aligned}$$

Therefore, $\boldsymbol{A}$ is self-adjoint with respect to the weighted inner product. A similar argument reveals that

$$\boldsymbol{A} = \operatorname{diag}(\boldsymbol{p})^{-1}(\boldsymbol{M} - \boldsymbol{D}).$$

This formula ensures that $\boldsymbol{A}$ is still nonnegative and irreducible. Indeed, multiplication by a positive diagonal matrix does not affect either property.

Why did we do all this reweighting? The reason is that we want to make $\boldsymbol{h}(\mathsf{P}_1)$ into an eigenvector of $\boldsymbol{A}$ with eigenvalue 1. Indeed,

$$(\boldsymbol{A}\boldsymbol{h}(\mathsf{P}_1))_i = \frac{1}{d+2} \cdot \frac{1}{p_i} \cdot V(\mathsf{F}_i(\mathsf{P}_1), \mathsf{F}_i(\mathsf{P}_1), \dots, \mathsf{F}_i(\mathsf{P}_d)) = h_i(\boldsymbol{P}_1).$$

We are going to prove that 1 is the only positive eigenvalue of $\boldsymbol{A}$.

First, let is prove that 1 is the largest eigenvalue of $\boldsymbol{A}$. Perron's theorem shows that the positive vector $\boldsymbol{h}(\mathsf{P}_1)$ is the unique eigenvector associated with the maximum eigenvalue of $\boldsymbol{A} + \alpha\mathbf{I}$, where $\alpha > 0$ is some very big number that we use to kill of potentially negative numbers on the diagonal. This in turn ensures that 1 is the maximum eigenvalue of $\boldsymbol{A}$, and it is a simple eigenvalue.

Next, we will prove that the remaining eigenvalues of $\boldsymbol{A}$ are nonpositive. This point follows from the following claim:

$$\langle \boldsymbol{A}\boldsymbol{x},\, \boldsymbol{A}\boldsymbol{x}\rangle_{\boldsymbol{p}} \geq \langle \boldsymbol{x},\, \boldsymbol{A}\boldsymbol{x}\rangle_{\boldsymbol{p}} \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^N. \tag{19.5.1}$$

For the moment, assume that (19.5.1) holds, and let us complete the proof.

Let $(\boldsymbol{x}, \lambda)$ be an eigenpair of $\boldsymbol{A}$. The claim (19.5.1) implies that the eigenvalue satisfies $\lambda^2 \geq \lambda$. This in turn demands $\lambda \geq 1$ or $\lambda \leq 0$. But 1 is the maximum eigenvalue of $\boldsymbol{A}$, and it is simple. Therefore, the matrix $\boldsymbol{A}$ has Lorentz signature (i.e., the matrix has at most one positive eigenvalue). Lemma 19.4.2 now ensures that

$$\langle \boldsymbol{x},\, \boldsymbol{A}\boldsymbol{x}\rangle_{\boldsymbol{p}}^2 \geq \langle \boldsymbol{x},\, \boldsymbol{A}\boldsymbol{x}\rangle_{\boldsymbol{p}} \cdot \langle \boldsymbol{h}(\mathsf{K}),\, \boldsymbol{A}\boldsymbol{h}(\mathsf{K})\rangle_{\boldsymbol{p}}.$$

Equivalently,

$$V(\boldsymbol{x}, \mathsf{K}, \mathsf{P}_1, \dots, \mathsf{P}_d)^2 \geq V(\boldsymbol{x}, \boldsymbol{x}, \mathsf{P}_1, \dots, \mathsf{P}_d) \cdot V(\mathsf{K}, \mathsf{K}, \mathsf{P}_1, \dots, \mathsf{P}_d).$$

This is what we needed to show.

Finally, let us verify the claim (19.5.1). This is one of the primary new insights that Shenfeld & Van Handel provided. Use the induction hypothesis to compute

$$\begin{aligned}(\boldsymbol{A}\boldsymbol{x})_i^2 p_i &= \frac{1}{d+2} \cdot h_i(\mathsf{P}_1) \cdot \frac{V(\mathsf{F}_i(\boldsymbol{x}), \mathsf{F}_i(\mathsf{P}_i), \dots, \mathsf{F}_i(\mathsf{P}_i))^2}{V(\mathsf{F}_i(\mathsf{P}_1), \mathsf{F}_i(\mathsf{P}_1), \dots, \mathsf{F}_i(\mathsf{P}_d))} \\ &\geq \frac{1}{d+2} \cdot h_i(\mathsf{P}_1) \cdot V(\mathsf{F}_i(\boldsymbol{x}), \mathsf{F}_i(\boldsymbol{x}), \mathsf{F}_i(\mathsf{P}_2), \dots, \mathsf{F}_i(\mathsf{P}_d)).\end{aligned}$$

Summing over the indices reveals

$$\langle \boldsymbol{A}\boldsymbol{x},\, \boldsymbol{A}\boldsymbol{x}\rangle_{\boldsymbol{p}} \geq \frac{1}{d+2} \sum\nolimits_{i=1}^{N} h_i(\mathsf{P}_1) \cdot V(\mathsf{F}_i(\boldsymbol{x}), \mathsf{F}_i(\boldsymbol{x}), \mathsf{F}_i(\mathsf{P}_2), \dots, \mathsf{F}_i(\mathsf{P}_d)) = \langle \boldsymbol{x},\, \boldsymbol{A}\boldsymbol{x}\rangle_{\boldsymbol{p}}.$$

The claim follows from induction.

# Problem Set 1

*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*10 October 2018*

## 20.1 Overview

This assignment covers basic definitions in affine geometry and convexity, operations that preserve convexity, topology of convex sets, combinatorial convexity, support and separation, faces, exposed faces, and the theorems of Minkowski and Dubins.

### 20.1.1 Directions

It may take a long time to do everything, so just do as much as you can and turn that in. Starred problems are optional; the number of stars reflects the difficulty. You are welcome to collaborate with your peers, but you must write up your own solutions. Please avoid books or the internet unless you are really stuck. If you use any resources to solve the problems, you must cite them in your solution. Please follow the Homework Guide when preparing your assignment.

## 20.2 Exercises

Exercises involve important definitions and basic facts that you should verify for yourself. These statements often play a role in other problems or in class. Do not write up the solutions to the exercises.

1. **(Preserving Convexity).** Let $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$ be (nonempty) convex sets.

   a) Show that the intersection $\mathsf{C} \cap \mathsf{K}$ and the direct product $\mathsf{C} \times \mathsf{K}$ are convex. Is the union $\mathsf{C} \cup \mathsf{K}$ convex?
   b) For $\lambda \in \mathbb{R}$, show that the dilation $\lambda\mathsf{C} := \{\lambda \boldsymbol{x} : \boldsymbol{x} \in \mathsf{C}\}$ is convex. Show that the Minkowski sum $\mathsf{C} + \mathsf{K} := \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in \mathsf{C}, \boldsymbol{y} \in \mathsf{K}\}$ is convex.
   c) Let $\boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^d$ be an affine map, i.e., the composition of a linear map and a translation. Show that the image $\boldsymbol{T}\mathsf{C}$ is convex. Show that the preimage $\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{T}\boldsymbol{x} \in \mathsf{C}\}$ is convex.

2. **(*A Characterization of Convexity).** Prove that $\mathsf{C}$ is convex if and only if $(\lambda + \mu)\mathsf{C} = \lambda\mathsf{C} + \mu\mathsf{C}$ for all $\lambda, \mu > 0$.

3. **(Convex Cones).** A *convex cone* is a convex set $\mathsf{K} \subset \mathbb{R}^d$ that is positively homogeneous: $\lambda\mathsf{K} = \mathsf{K}$ for $\lambda > 0$. With this definition, a convex cone is always "anchored" at zero. Which of the set operations in Exercise (1) preserve the property of positive homogeneity?

4. **(Topology).** In this exercise, we work out the basic topological properties of convex hulls.

   a) Let $\mathsf{C}$ be a (nonempty) convex set. Let $\boldsymbol{x} \in \operatorname{bd} \mathsf{C}$ and $\boldsymbol{y} \in \operatorname{int} \mathsf{C}$. Prove that the open segment $(\boldsymbol{x}, \boldsymbol{y}) \subset \operatorname{int} \mathsf{C}$.

b) Show that the convex hull of an open set is open.
c) Show that the convex hull of a compact set is compact. **Hint:** Carathéodory.
d) Do any/all of the operations in Exercise (1) preserve topological properties (openness, closedness, compactness)?
e) Prove in detail that a simplex (i.e., the convex hull of affinely independent points) contains a point in its relative interior.

5. **(Distance).** Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty, closed, convex set. Compute the gradient of the map $\boldsymbol{x} \mapsto \frac{1}{2}\operatorname{dist}^2(\boldsymbol{x}, \mathsf{C})$.

6. **(*Extreme Points).** By a direct argument (not using Minkowski), prove that a compact, convex set in $\mathbb{R}^d$ has an extreme point.

## 20.3 Problems

Problems require more substantial investigations. You will write up the solutions to the problems.

1. **(Radon + Helly).** Carathéodory's theorem is one of the three fundamental results in combinatorial convexity. This problem concerns the other two results and a geometric application.

   a) **(Radon).** Prove the Radon theorem: Let $\mathsf{A} \subset \mathbb{R}^d$ be a finite, affinely *dependent* set. Then we can partition $\mathsf{A} = \mathsf{B} \cup \mathsf{R}$ into disjoint sets (i.e., $\mathsf{B} \cap \mathsf{R} = \emptyset$) whose convex hulls overlap:
   $$\operatorname{conv} \mathsf{B} \cap \operatorname{conv} \mathsf{R} \neq \emptyset.$$
   **Hint:** In the affine dependency condition, collect points with positive coefficients together.
   b) **(Helly).** Prove the Helly theorem: Let $\mathsf{C}_1, \dots, \mathsf{C}_n$ be convex sets in $\mathbb{R}^d$, where $n \geq d+1$. Suppose that each subfamily of $d+1$ convex sets has a nonempty intersection. Then the whole family of convex sets has a nonempty intersection. **Hints:** For fixed $d$, use induction on $n$. The case $n = d+2$ contains the main idea: apply Radon's theorem to points drawn from all intersections of subfamilies of $d+1$ convex sets.
   c) **(*Helly $\infty$).** Extend Helly's theorem from a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$ to an *arbitrary* family of at least $d+1$ *compact* convex sets in $\mathbb{R}^d$. Produce a family of noncompact convex sets where the conclusion fails.
   d) **(*Vincensini).** Prove Vincensini's extension of Helly's theorem: Let $\mathsf{C}_1, \dots, \mathsf{C}_n$ be convex sets in $\mathbb{R}^d$, where $n \geq d+1$. Suppose that each subfamily of $d+1$ convex sets contains a translate of a fixed convex set $\mathsf{K} \subset \mathbb{R}^d$. Prove that the intersection of the entire family contains a translate of $\mathsf{K}$. Establish "Vincensini $\infty$," in the spirit of (c).
   e) **(Strips).** In $\mathbb{R}^2$, a (closed) strip of width $w$ is the locus of points contained between two parallel lines at distance $w$. The *minimum width* of a set is the smallest width of a strip that contains the set. Prove that each compact convex set of minimum width one contains a segment of length one in each direction. **Hint:** Dual representation via halfspaces!

2. **(Unbounded Minkowski).** We shall extend Minkowski's theorem to closed, but unbounded, convex sets. Let $\mathsf{C} \subset \mathbb{R}^d$ be a nonempty closed convex set.

   a) Prove that the *recession cone*, $\operatorname{rec} \mathsf{C}$, is a closed convex cone:
   $$\operatorname{rec} \mathsf{C} := \{\boldsymbol{u} \in \mathbb{R}^d : \boldsymbol{u} + \mathsf{C} \subset \mathsf{C}\}.$$
   b) (*) Check that the *lineality space*, $\operatorname{lin} \mathsf{C}$, is a linear subspace:
   $$\operatorname{lin} \mathsf{C} := \operatorname{rec} \mathsf{C} \cap (-\operatorname{rec} \mathsf{C}).$$
   We say that a closed convex set is *line-free* if its lineality space is the trivial subspace $\{\mathbf{0}\}$.
   c) (*) Prove that $\mathsf{C} = (\operatorname{lin} \mathsf{C}) + \mathsf{C}_0$, where $\mathsf{C}_0$ is a closed, convex, line-free set contained in the orthogonal complement of $\operatorname{lin} \mathsf{C}$. Therefore, we may as well ignore the lineality space.
   d) Show that a closed convex $\mathsf{K}$ satisfies $\mathsf{K} = \operatorname{conv} \operatorname{relbd} \mathsf{K}$ unless $\mathsf{K}$ is an affine space or a halfspace of an affine space. **Hint:** Prove by contradiction, using weak separation.
   e) An *extreme ray* of a closed convex set $\mathsf{C}$ is a face $\{\boldsymbol{x} + \lambda \boldsymbol{s} : \lambda \geq 0\}$ where $\boldsymbol{x} \in \mathsf{C}$ and $\boldsymbol{s} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. The set of extreme rays is $\operatorname{extr} \mathsf{C}$. For a face $\mathsf{F}$ of $\mathsf{C}$, check that $\operatorname{extr} \mathsf{F} \subset \operatorname{extr} \mathsf{C}$.
   f) Prove that a closed, convex, line-free set $\mathsf{K}$ admits the decomposition
   $$\mathsf{K} = \operatorname{conv}(\operatorname{ext} \mathsf{K} \cup \operatorname{extr} \mathsf{K}).$$
   **Hint:** Mimic the proof of Minkowski's theorem, using the foregoing results.
   g) (*) Conclude that a closed, convex, line-free set $\mathsf{K}$ has an extreme point and, moreover,
   $$\mathsf{K} = (\operatorname{conv} \operatorname{ext} \mathsf{K}) + (\operatorname{rec} \mathsf{K}).$$
   **Hint:** A ray that is not a line has an extreme point.

3. **(Uncertainty Quantification).** Minkowski's theorem and Dubins's theorem have appealing applications in probability. In this problem, we will develop some finite-dimensional examples. Let $\mathsf{S} \subset \mathbb{R}$ be a finite set. Consider the collection of probability distributions on $\mathsf{S}$:
$$\Delta(\mathsf{S}) := \left\{\boldsymbol{p} : \mathsf{S} \to \mathbb{R}_+ : \sum\nolimits_{s \in \mathsf{S}} p(s) = 1 \text{ and } p(s) \geq 0 \text{ for all } s \in \mathsf{S}\right\} \subset \mathbb{R}^{\mathsf{S}}$$
We equip $\mathbb{R}^{\mathsf{S}}$ with the canonical inner product. Let $X$ be a random variable with values in $\mathsf{S}$.

   a) Show that $\Delta(\mathsf{S})$ is compact and convex. Identify all of the extreme points, with proof.
   b) Report all optimal solutions to the uncertainty quantification problem
   $$\mathsf{maximize}_X \quad \mathbb{E}[X^2]$$

c) Report at least one optimal solution to the uncertainty quantification problem

$$\text{maximize}_X \quad \operatorname{Var}[X]$$

**Hint:** Constrain $\mathbb{E}\, X = \alpha$ for a fixed $\alpha \in \mathbb{R}$.

d) (*) For $\alpha \in \mathbb{R}$, report an one optimal solution to the uncertainty quantification problem

$$\text{maximize}_X \quad \mathbb{P}\{X \geq s\} \quad \text{subject to} \quad \mathbb{E}\, X = \alpha.$$

e) (*) Show that a standard-form linear program with $m$ equality constraints has a solution with at most $m + 1$ nonzero entries.

4. **(Semidefinite Programming).** Dubins's theorem and its relatives have incredible consequences. A case of particular importance involves the positive-semidefinite (psd) cone. We work in the complex setting because the computations are cleaner. The set of Hermitian matrices of order $n$ is

$$\mathbb{H}^n := \{\boldsymbol{A} \in \mathbb{C}^{n \times n} : \boldsymbol{A} = \boldsymbol{A}^*\}.$$

We equip the Hermitian matrices with the trace inner product:

$$\langle \boldsymbol{A},\, \boldsymbol{B} \rangle := \operatorname{trace}(\boldsymbol{AB}) \quad \text{for } \boldsymbol{A}, \boldsymbol{B} \in \mathbb{H}^n.$$

The complex psd cone is

$$\mathbb{H}^n_+ := \{\boldsymbol{A} \in \mathbb{H}^n : \boldsymbol{u}^* \boldsymbol{A} \boldsymbol{u} \geq 0 \text{ for all } \boldsymbol{u} \in \mathbb{C}\}.$$

The complex positive-definite (pd) cone is

$$\mathbb{H}^n_{++} := \{\boldsymbol{A} \in \mathbb{H}^n : \boldsymbol{u}^* \boldsymbol{A} \boldsymbol{u} > 0 \text{ for all } \textit{nonzero } \boldsymbol{u} \in \mathbb{C}\}.$$

The main result of this problem is due, independently, to Barvinok (1995) and Pataki (1998).

a) Verify that $\mathbb{H}^n$ is a *real* linear space with dimension $n^2$.

b) Explain why $\mathbb{H}^n_+$ is a line-free, closed, convex cone. Check that $\mathbb{H}^n_{++}$ is the interior of $\mathbb{H}^n_+$.

c) Let $\boldsymbol{A} \in \mathbb{H}^n_+$ be a psd matrix with rank $r < n$. Show that $\boldsymbol{A}$ is an internal point of an exposed face $\mathsf{F} \subset \mathbb{H}^n_+$ with dimension $r^2$. **Hint:** Consider the linear functional $\varphi = \langle \boldsymbol{P},\, \cdot \rangle$ where $\boldsymbol{P}$ is the orthogonal projector onto null $\boldsymbol{A}$. In fact, $\mathsf{F}$ is essentially a copy of $\mathbb{H}^r_+$.

d) (*) Confirm that the faces of $\mathbb{H}^n_+$ are in one-to-one correspondence with subspaces of $\mathbb{C}^n$. Show that the correspondence reverses inclusion.

e) Let $\mathsf{L}$ be a nontrivial affine space with codimension $m$, and assume $\mathsf{E} := \mathsf{H}^n_+ \cap \mathsf{L}$ is nonempty. Show that each extreme point of $\mathsf{E}$ is contained in the relative interior of an exposed face $\mathsf{F}$ of $\mathsf{H}^n_+$ with dimension at most $m$. **Hint:** We must have $\dim(\mathsf{F} \cap \mathsf{L}) = 0$.

f) Conclude that the extreme points of $\mathsf{E}$ include a matrix with rank at most $\sqrt{m}$.

g) (*) Reinterpret: A standard-form SDP with $m$ equality constraints has a solution whose rank is at most $\sqrt{m}$.

h) (*) Describe the proper faces of the set $\mathsf{C} := \{\boldsymbol{A} \in \mathbb{H}_n^+ : \operatorname{trace} \boldsymbol{A} = 1\}$ of quantum states. With $\mathsf{L}$ as before, prove the extreme points of $\mathsf{C} \cap \mathsf{L}$ include a matrix with rank at most $\sqrt{m}$.

# Problem Set 2

*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*29 October 2018*

## 21.1 Overview

This assignment covers polytopes, polyhedra, normal cones, polarity, the Weyl–Minkowski theorem, Hausdorff distance, support functions, Steiner's formula, intrinsic volumes, valuations, and the Euler characteristic.

### 21.1.1 Directions

It may take a long time to do everything, so just do as much as you can and turn that in. Starred problems are optional; the number of stars reflects the difficulty. You are welcome to collaborate with your peers, but you must write up your own solutions. Please avoid books or the internet unless you are really stuck. If you use any resources to solve the problems, you must cite them in your solution. Please follow the Homework Guide when preparing your assignment.

## 21.2 Exercises

Exercises involve important definitions and basic facts that you should verify for yourself. These statements often play a role in other problems or in class. Do not write up the solutions to the exercises.

1. **(Continuity of set operations).** Some basic operations on compact convex sets are continuous with respect to the Hausdorff metric. Suppose that $\{\mathsf{C}_i : i \in \mathbb{N}\} \subset \mathbb{R}^d$ is a sequence of nonempty convex bodies that converges in Hausdorff metric to a nonempty convex body $\mathsf{C} \subset \mathbb{R}^d$. That is, $\mathsf{C}_i \to \mathsf{C}$. Let $\mathsf{K} \subset \mathbb{R}^d$ be a fixed convex body.

    a) Confirm that the Hausdorff distance $\operatorname{dist}_{\mathrm{H}}$ is a metric on convex bodies.
    b) Show that Minkowski addition is continuous: $\mathsf{C}_i + \mathsf{K} \to \mathsf{C} + \mathsf{K}$.
    c) Show that an affine map $\boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^m$ is continuous: $\boldsymbol{T}\mathsf{C}_i \to \boldsymbol{T}\mathsf{C}$.
    d) (*) Show that intersection may be *discontinuous*: Sometimes, $\mathsf{C}_i \cap \mathsf{K} \not\to \mathsf{C} \cap \mathsf{K}$. Show that continuity of the intersection is restored if $\mathsf{C}$ and $\mathsf{K}$ cannot be separated by a hyperplane.

2. **(Polytope calculus).** Let us verify some key facts about polytopes and polyhedra. These are not hard, once we have polarity and the Weyl–Minkowski theorem at hand.

    a) For sets $\mathsf{A}, \mathsf{B}$, check the following polar relations:

    $$(\mathsf{A} \cup \mathsf{B})^\circ = \mathsf{A}^\circ \cap \mathsf{B}^\circ \quad \text{and} \quad (\mathsf{A} \cap \mathsf{B})^\circ = \operatorname{conv}(\mathsf{A}^\circ \cup \mathsf{B}^\circ).$$

    b) Prove that the Minkowski sum of two polytopes is a polytope.
    c) Prove that the intersection of two polytopes is a polytope.
    d) Check that a face of a polytope is a polytope, and a polytope has finitely many faces.

e) Check that a face of a polyhedron is a polyhedron; a polyhedron has finitely many faces.
f) Prove that an affine slice of a polytope is a polytope.

3. **(Valuations).** Show that indicator functions satisfy the inclusion–exclusion law:

$$[\mathsf{A}_1 \cup \cdots \cup \mathsf{A}_m] = \sum_{k=1}^{m} (-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} [\mathsf{A}_{i_1} \cap \cdots \cap \mathsf{A}_{i_k}].$$

What is the $m = 2$ case? Conclude that a linear valuation $\varphi$ satisfies

$$\varphi([\mathsf{A}_1 \cup \cdots \cup \mathsf{A}_m]) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} \varphi([\mathsf{A}_{i_1} \cap \cdots \cap \mathsf{A}_{i_k}]).$$

(**) Does a set valuation always satisfy the analogous relation?

## 21.3 Problems

Problems require more substantial investigations. You will write up the solutions to the problems.

1. **(Conjugate Faces).** Let $\mathsf{P} \subset \mathbb{R}^d$ be a polytope that contains the origin in its interior, and let $\mathsf{P}^\circ$ be its polar. In this problem, we will establish some important facts about the facial structure of the polytope and its polar. For a face $\mathsf{F} \lhd \mathsf{P}$, define the *conjugate face* $\mathsf{F}^\diamond$ of $\mathsf{P}^\circ$ as

$$\mathsf{F}^\diamond := \{\boldsymbol{s} \in \mathsf{P}^\circ : \langle \boldsymbol{s},\, \boldsymbol{x} \rangle = 1 \text{ for all } \boldsymbol{x} \in \mathsf{F}\}.$$

We write $\lhd$ for the "face of" relation.

a) If $\mathsf{F} \lhd \mathsf{P}$, show that $\mathsf{F}^\diamond \lhd \mathsf{P}^\circ$.
b) Prove that $\dim \mathsf{F} + \dim \mathsf{F}^\diamond = d - 1$.
c) A *facet* of a polytope is a face with dimension $d - 1$. Show that every polytope has a facet. Deduce that a $d$-dimensional polytope has a face of every dimension $j = 0, 1, 2, \dots, d$.
d) Show that every polytope is the intersection of the halfspaces determined by its facets:

$$\mathsf{P} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s}_i,\, \boldsymbol{x} \rangle \le \alpha_i \text{ for } i = 1, \dots, m\},$$

where $\mathsf{F}_i = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s}_i,\, \boldsymbol{x} \rangle = \alpha_i\}$ is a facet of $\mathsf{P}$ for each $i$. Conclude that every proper face of a polytope is the intersection of the facets that contain it.
e) (*) Check that facial conjugacy reverses inclusion: If $\mathsf{F}, \mathsf{G} \lhd \mathsf{P}$, then $\mathsf{F} \subset \mathsf{G}$ implies $\mathsf{G}^\diamond \subset \mathsf{F}^\diamond$.
f) (*) If $\mathsf{F} \lhd \mathsf{P}$, show that $\mathsf{F}^{\diamond\diamond} = \mathsf{F}$. Conclude that $\diamond$ is a bijection between faces of $\mathsf{P}$ and $\mathsf{P}^\circ$.
g) (*) Show that $\mathsf{F}^\diamond$ is a base of the normal cone $\mathsf{N}(\mathsf{F})$. That is, $\mathsf{N}(\mathsf{F}) = \operatorname{cone} \mathsf{F}^\diamond$ and each ray of $\mathsf{N}(\mathsf{F})$ pierces $\mathsf{F}^\diamond$ in exactly one point.

2. **(Pyramids).** Let $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$ be closed convex sets, and let $h(\cdot; \mathsf{C})$ denote the support function of the set $\mathsf{C}$.

a) Check that $h(\boldsymbol{s}; \mathsf{C} + \mathsf{K}) = h(\boldsymbol{s}; \mathsf{C}) + h(\boldsymbol{s}; \mathsf{K})$ for each $\boldsymbol{s} \in \mathbb{R}^d$.

b) Let $\mathsf{E} \subset \mathbb{R}^{d-1}$ be a closed convex set, and fix $a > 0$. A *pyramid* is a set of the form
$$\mathsf{T} := \{\bar{\tau} \cdot (\boldsymbol{x}, 0) + \tau \cdot (\mathbf{0}, a) : \boldsymbol{x} \in \mathsf{E} \text{ and } \tau \in [0,1]\} \subset \mathbb{R}^d.$$
Sketch a few pyramids in $\mathbb{R}^2$ and $\mathbb{R}^3$. Then prove that the volume of the pyramid $\mathsf{T}$ is
$$\mathrm{Vol}_d(\mathsf{T}) = \frac{1}{d} \cdot \mathrm{Vol}_{d-1}(\mathsf{E}) \cdot a.$$
Compare with the formulas for the area of a triangle and volume of a circular cone.

c) (*) Use the latter expression recursively to determine the volume of the set
$$\mathsf{T} = \left\{ \boldsymbol{x} \in \mathbb{R}^d_+ : \sum\nolimits_{i=1}^d x_i \leq 1 \right\}.$$
Obtain, as a consequence, the volume of the $\ell_1^d$ unit ball. **Hint:** Draw some pictures!

d) More generally, consider a bounded polyhedron that contains the origin in its interior, with $m$ facets, given explicitly as
$$\mathsf{P} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \leq \alpha_i \text{ for } i = 1, \dots, m\} \quad \text{where } \|\boldsymbol{u}_i\| = 1 \text{ for each } i.$$
Let $\mathsf{F}_i$ be the facet with outer normal $\boldsymbol{u}_i$. Check that $h(\boldsymbol{u}_i; \mathsf{P}) = \alpha_i > 0$ for each $i$, and
$$\mathrm{Vol}_d(\mathsf{P}) = \frac{1}{d} \sum\nolimits_{i=1}^m \mathrm{Vol}_{d-1}(\mathsf{F}_i) \cdot h(\boldsymbol{u}_i; \mathsf{P}).$$

e) For a unit vector $\boldsymbol{z} \in \mathbb{R}^d$, relate the surface area of the facet $\mathsf{F}_i$ to the surface area of its orthogonal projection $\mathsf{F}_i \,|\, \boldsymbol{z}^\perp$ onto the hyperplane $\boldsymbol{z}^\perp$:
$$\mathrm{Vol}_{d-1}(\mathsf{F}_i \,|\, \boldsymbol{z}^\perp) = \mathrm{Vol}_{d-1}(\mathsf{F}_i) \cdot |\langle \boldsymbol{u}_i, \boldsymbol{z} \rangle|.$$

f) (*) Confirm the identity
$$\sum\nolimits_{i=1}^m \mathrm{Vol}_{d-1}(\mathsf{F}_i) \cdot \langle \boldsymbol{u}_i, \boldsymbol{z} \rangle = 0 \quad \text{for all nonzero } \boldsymbol{z} \in \mathbb{R}^d.$$
Check that the formula in (e) for $\mathrm{Vol}_d(\mathsf{P})$ is still valid when $\mathsf{P}$ does not contain the origin.

3. **(Steiner, Wills, Cauchy, and Kubota).** In this problem, we will develop some extensions and applications of Steiner's formula, including alternative presentations of the intrinsic volumes.

a) Use Steiner's formula to compute the intrinsic volumes of the Euclidean ball $\mathsf{B}_d$.

b) For a convex body $\mathsf{C} \subset \mathbb{R}^d$, prove Wills's formula:
$$W(\mathsf{C}) := \int_{\mathbb{R}^d} \mathrm{e}^{-\pi \,\mathrm{dist}^2(\boldsymbol{x}; \mathsf{C})} \,\mathrm{d}\boldsymbol{x} = \sum\nolimits_{j=0}^d V_j(\mathsf{C}).$$

c) (*) Wills's formula yields the intrinsic volumes of a direct product of convex bodies:

$$V_j(\mathsf{C} \times \mathsf{K}) = \sum\nolimits_{i+k=j} V_i(\mathsf{C}) \cdot V_k(\mathsf{K}). \quad \text{for } j = 0, 1, 2, \dots, d.$$

**Hint:** Compute $W(\lambda(\mathsf{C} \times \mathsf{K}))$ for nonnegative $\lambda$.

d) (*) A parallelotope is the direct product of line segments: $[0, s_1] \times \cdots \times [0, s_n]$. Compute the intrinsic volumes of the parallelotope in terms of the side lengths $s_i$.

e) Prove Steiner's formula for intrinsic volumes:

$$V_j(\mathsf{C} + \lambda \mathsf{B}_d) = \sum\nolimits_{i \le j} \lambda^{j-i} \cdot \frac{\kappa_{d-i}}{\kappa_{d-j}} \binom{d-i}{d-j} \cdot V_i(\mathsf{C}).$$

The number $\kappa_i := \mathrm{Vol}_i(\mathsf{B}_i)$. **Hint:** Express the volume of $\mathsf{C} + (\lambda + \mu)\mathsf{B}_d$ in two ways.

f) For a polytope $\mathsf{P} \subset \mathbb{R}^d$, explain why $2V_{d-1}(\mathsf{P})$ equals the surface area of $\mathsf{P}$. Then establish Cauchy's surface area formula:

$$V_{d-1}(\mathsf{P}) = \frac{1}{2\kappa_{d-1}} \int_{\mathbb{S}^{d-1}} V_{d-1}(\mathsf{P} \,|\, \boldsymbol{u}^\perp) \, \mathrm{d}\sigma(\boldsymbol{u})$$

We have written $\mathsf{P} \,|\, \boldsymbol{u}^\perp$ for the orthogonal projection onto the hyperplane $\boldsymbol{u}^\perp$ and $\mathrm{d}\sigma$ for the surface measure on the sphere (induced by the Lebesgue measure).

g) Explain how to extend Cauchy's formula from polytopes to convex bodies.

h) (*) Kubota showed that the other intrinsic volumes admit Cauchy-like formulas:

$$V_j(\mathsf{C}) = \frac{\kappa_{d-1-j}}{(d-j)\kappa_{d-j}\kappa_{d-1}} \int_{\mathbb{S}^{d-1}} V_j(\mathsf{C} \,|\, \boldsymbol{u}^\perp) \, \mathrm{d}\sigma(\boldsymbol{u}) \quad \text{for } j = 0, 1, 2, \dots, d-1.$$

Prove it. **Hints:** Apply Steiner's formula to compute $V_{d-1}((\mathsf{C} + \lambda \mathsf{B}_d) \,|\, \boldsymbol{u}^\perp)$. You'll also need Cauchy's formula and Steiner's formula for intrinsic volumes.

i) (**) Kubota also extended Cauchy's formula to higher-dimensional projections:

$$V_j(\mathsf{C}) = \frac{\kappa_d}{\kappa_j \kappa_{d-j}} \binom{d}{j} \int_{G(j,d)} V_j(\mathsf{C} \,|\, \mathsf{L}) \, \mathrm{d}\nu_j(\mathsf{L})$$

where $\mathsf{C} \,|\, \mathsf{L}$ is the orthogonal projection of $\mathsf{C}$ onto a $j$-dimensional subspace $\mathsf{L}$ and $\nu_j$ is the rotation-invariant probability measure on the family $G(j, d)$ of $j$-dimensional subspaces of $\mathbb{R}^d$. **Hint:** We can construct a uniformly random subspace by projecting out one uniformly random direction at a time.

j) Specialize Kubota's projection formula to the case $j = 1$. How does this result help us interpret the intrinsic volume $V_1$? What does Kubota's formula signify for other $j$?

k) Use Kubota's projection formula to confirm that intrinsic volumes are monotone with respect to inclusion. That is, $\mathsf{C} \subset \mathsf{C}'$ implies that $V_j(\mathsf{C}) \le V_j(\mathsf{C}')$ for each index $j$.

4. **(*Conic Geometry).** In this problem, we explore the elegant duality theory of convex cones. We will also establish Moreau's theorem, a very important fact about convex cones. It can be viewed as an analog of the Peirce decomposition of the identity into two orthogonal projectors onto complementary subspaces. Let $\mathsf{K}, \mathsf{K}_1, \mathsf{K}_2 \subset \mathbb{R}^d$ be nonempty, closed, convex cones.

   a) Define the polar cone
   $$\mathsf{K}^\circ := \{\boldsymbol{s} \in \mathbb{R}^d : \langle \boldsymbol{s},\, \boldsymbol{x} \rangle \leq 0 \text{ for all } \boldsymbol{x} \in \mathsf{K}.\}.$$
   Show that this definition coincides with the definition in Lecture 5, where zero is replaced by one. Prove that the polar cone is a nonempty, closed, convex cone.
   b) What is the polar cone of a linear subspace? What is the polar cone of the nonnegative orthant? What is the polar cone of the set of positive-semidefinite matrices?
   c) Specialize the variational characterization of a projector to the conic setting. For $\boldsymbol{x} \in \mathbb{R}^d$,
   $$\boldsymbol{y} = \mathrm{proj}_{\mathsf{K}}(\boldsymbol{x}) \quad \text{if and only if} \quad \boldsymbol{y} \in \mathsf{K}, \quad \boldsymbol{x} - \boldsymbol{y} \in \mathsf{K}^\circ, \quad \langle \boldsymbol{x} - \boldsymbol{y},\, \boldsymbol{y} \rangle = 0.$$
   d) Establish the following properties of the projector onto a cone. For $\boldsymbol{x} \in \mathbb{R}^d$,
      i. $\mathrm{proj}_{\mathsf{K}}(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} \in \mathsf{K}^\circ$.
      ii. $\mathrm{proj}_{\mathsf{K}}(\lambda \boldsymbol{x}) = \lambda \mathrm{proj}_{\mathsf{K}}(\boldsymbol{x})$ for $\lambda \geq 0$.
      iii. $\mathrm{proj}_{\mathsf{K}}(-\boldsymbol{x}) = -\mathrm{proj}_{-\mathsf{K}}(\boldsymbol{x})$.
      iv. $\boldsymbol{x} = \boldsymbol{p}_{\mathsf{K}}(\boldsymbol{x}) + \boldsymbol{p}_{\mathsf{K}^\circ}(\boldsymbol{x})$.
   e) Prove Moreau's theorem: Each point $\boldsymbol{x} \in \mathbb{R}^d$ has an orthogonal decomposition $\boldsymbol{x} = \boldsymbol{k} + \boldsymbol{n}$ where $\boldsymbol{k} \in \mathsf{K}$ and $\boldsymbol{n} \in \mathsf{K}^\circ$ and $\langle \boldsymbol{k},\, \boldsymbol{n} \rangle = 0$.
   f) Use Moreau's theorem to prove that every symmetric matrix can be represented uniquely as the difference of two psd matrices.
   g) Establish the formulas
   $$(\mathsf{K}_1 \cap \mathsf{K}_2)^\circ = \mathsf{K}_1^\circ + \mathsf{K}_2^\circ \quad \text{and} \quad (\mathsf{K}_1 + \mathsf{K}_2)^\circ = \mathsf{K}_1^\circ \cap \mathsf{K}_2^\circ.$$
   Show that polarity reverses inclusion: $\mathsf{K}_1 \subset \mathsf{K}_2$ implies $\mathsf{K}_2^\circ \subset \mathsf{K}_1^\circ$.
   h) Prove the bipolar theorem: $(\mathsf{K}^\circ)^\circ = \mathsf{K}$.
   i) A cone is *polyhedral* if it is a finite intersection of halfspaces containing the origin. Prove that a polyhedral cone has a finite number of extreme rays.
   j) (**) Let $\mathsf{K} \subset \mathbb{R}^d$ be a polyhedral cone. For a bounded continuous function $f : \mathbb{R}_+^2 \to \mathbb{R}$, define
   $$\varphi_f(\mathsf{K}) := \mathbb{E}\left[ f\big( \|\mathrm{proj}_{\mathsf{K}}(\boldsymbol{g})\|^2, \|\mathrm{proj}_{\mathsf{K}^\circ}(\boldsymbol{g})\|^2 \big) \right] \quad \text{where } \boldsymbol{g} \sim \textsc{normal}(\mathbf{0}, \mathbf{I}).$$
   Prove the master Steiner formula for polyhedral cones:
   $$\varphi_f(\mathsf{K}) = \sum\nolimits_{j=0}^{d} \varphi_f(\mathsf{L}_j) \cdot v_j(\mathsf{K}).$$
   where $\mathsf{L}_j$ is a $j$-dimensional subspace. The numbers $v_j$ are called *conic intrinsic volumes*.
   k) (**) Extend the conic Steiner formula to all cones and all Borel measurable functions for which the expectations are finite.

# Problem Set 3

*ACM 204, Fall 2018*
*Prof. Joel A. Tropp*
*20 November 2018*

## 22.1 Overview

This assignment covers valuations, the Euler characteristic, integral geometry, the Brunn–Minkowksi theorem, the Prékopa–Leindler inequality, Steiner symmetrization, the isoperimetric inequality, John's theorem, the reverse isoperimetric inequality, and the Brascamp–Lieb inequality.

### 22.1.1 Directions

It may take a long time to do everything, so just do as much as you can and turn that in. Starred problems are optional; the number of stars reflects the difficulty. You are welcome to collaborate with your peers, but you must write up your own solutions. Please avoid books or the internet unless you are really stuck. If you use any resources to solve the problems, you must cite them in your solution. Please follow the Homework Guide when preparing your assignment.

## 22.2 Exercises

Exercises involve important definitions and basic facts that you should verify for yourself. These statements often play a role in other problems or in class. Do not write up the solutions to the exercises.

1. **(Planar Kinematics).** Compute the measure $\mu$ of the set of rigid motions $\boldsymbol{T}$ that bring two convex sets $\mathsf{C}, \mathsf{K}$ in the plane into contact: $\mu\{\boldsymbol{T} : \mathsf{C} \cap \boldsymbol{T}\mathsf{K} \neq \emptyset\}$. Are any of these results obvious? Draw some pictures!

   a) $\mathsf{C} = \mathsf{B}_2$ and $\mathsf{K} = \mathsf{B}_2$.
   b) $\mathsf{C} = [0,1]^2$ and $\mathsf{K} = \mathsf{B}_2$.
   c) $\mathsf{C} = [0,1]^2$ and $\mathsf{K} = [0,1]^2$.
   d) $\mathsf{C} = [0,1]^2$ and $\mathsf{K} = [0,1] \times \{0\}$.

2. **(Brunn–Minkowski).** Prove that the following results are equivalent if they hold for all measurable sets $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$ and all $\tau \in [0,1]$, with $\bar{\tau} := 1 - \tau$.

   a) $\mathrm{Vol}_d(\mathsf{C} + \mathsf{K})^{1/d} \geq \mathrm{Vol}_d(\mathsf{C})^{1/d} + \mathrm{Vol}_d(\mathsf{K})^{1/d}$.
   b) $\mathrm{Vol}_d(\bar{\tau}\mathsf{C} + \tau\mathsf{K})^{1/d} \geq \bar{\tau}\,\mathrm{Vol}_d(\mathsf{C}^{1/d}) + \tau\,\mathrm{Vol}_d(\mathsf{K})^{1/d}$.
   c) $\mathrm{Vol}_d(\bar{\tau}\mathsf{C} + \tau\mathsf{K}) \geq \mathrm{Vol}_d(\mathsf{C})^{\bar{\tau}}\,\mathrm{Vol}_d(\mathsf{K})^{\tau}$.
   d) $\mathrm{Vol}_d(\bar{\tau}\mathsf{C} + \tau\mathsf{K}) \geq \min\{\mathrm{Vol}_d(\mathsf{C}), \mathrm{Vol}_d(\mathsf{K})\}$.

3. **(*Hurwitz).** There is an easy proof of the isoperimetric theorem in the plane, if you know a little Fourier analysis. Let $s \mapsto (x(s), y(s))$ for $s \in [0, 2\pi]$ be an arclength parameterization of a smooth, nonintersecting, closed curve in the plane. In other words, $(x(s), y(s))$ is the point at arclength $Ls/(2\pi)$ from $(x(0), y(0))$.

a) First, show that

$$\frac{1}{2\pi}\int_0^{2\pi}\left[x'(s)^2+y'(s)^2\right]\mathrm{d}s = L^2.$$

b) For a counterclockwise parameterization, show that the total area enclosed by the curve is

$$A = -\int_0^{2\pi} y(s)\,x'(s)\,\mathrm{d}s.$$

c) Expand $x$ and $y$ in Fourier series, and rewrite the integrals using Parseval's theorem.

d) Conclude that $L^2 \geq 4\pi A$, with equality if and only if the curve is a circle.

e) (**) Extend this result to all nonintersecting, closed plane curves.

## 22.3 Problems

Problems require more substantial investigations. You will write up the solutions to the problems.

1. **(Dual Volume).** Consider the class $\mathscr{C}_d^\circ$ of convex bodies in $\mathbb{R}^d$ that contain the origin in their interior, along with the empty set. We define $\emptyset^\circ = \emptyset$, and we assume $\mathsf{C}, \mathsf{K} \in \mathscr{C}_d^\circ$.

   a) Let $\mu$ be a set valuation on $\mathscr{C}_d^\circ$. Prove that the map $\mathsf{C} \mapsto \mu(\mathsf{C}^\circ)$ is a set valuation.

   b) Show that polarity is a Hausdorff continuous map on bodies in $\mathscr{C}_d^\circ$ that contain a fixed ball about the origin.

   c) Conclude that the dual volume $\mathrm{Vol}_d^\circ : \mathsf{C} \mapsto \mathrm{Vol}_d(\mathsf{C}^\circ)$ is a continuous set valuation on $\mathscr{C}_d^\circ$. (*) Extend the dual volume to a linear valuation on the algebra $\mathbb{A}(\mathscr{C}_d^\circ)$; discuss.

   d) Recall that $h(\boldsymbol{s};\mathsf{C})$ is the support function. Establish the formula

   $$\mathrm{Vol}_d^\circ(\mathsf{C}) = \frac{1}{d!}\int_{\mathbb{R}^d} \mathrm{e}^{-h(\boldsymbol{s};\mathsf{C})}\,\mathrm{d}\boldsymbol{s}.$$

   **Hint:** Integrate $\mathbb{1}_{\mathsf{C}^\circ}$ in spherical coordinates. (*) What is the analog for $\mathrm{Vol}_d$? Do these formulas extend to the algebra? In what form?

   e) Deduce Firey's inequality: For $\tau \in [0,1]$,

   $$\log \mathrm{Vol}_d^\circ((1-\tau)\mathsf{C} + \tau\mathsf{K}) \leq (1-\tau)\log \mathrm{Vol}_d^\circ(\mathsf{C}) + \tau \log \mathrm{Vol}_d^\circ(\mathsf{K}).$$

   That is, the dual volume is log-convex.

   f) (*) Formulate and derive an isoperimetric inequality that relates dual volume and "dual surface area."

2. **(Brunn–Minkowski).** In this problem, we will examine some of the consequences of the Brunn–Minkowski inequality.

a) Prove Brunn's slicing theorem: Let $\mathsf{C} \subset \mathbb{R}^d$ be a convex body, and fix a direction $\boldsymbol{s} \in \mathbb{R}^d$. Let $\mathsf{H}_\alpha := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s}, \boldsymbol{x} \rangle = \alpha\}$. Prove that

$$\alpha \mapsto \log \mathrm{Vol}_{d-1}(\mathsf{C} \cap \mathsf{H}_\alpha) \quad \text{is concave.}$$

Explain what this means, and draw a picture.

b) For $d$-dimensional convex bodies $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$, define the $\mathsf{K}$-surface area of $\mathsf{C}$ via

$$d \cdot S_{d-1}(\mathsf{C}; \mathsf{K}) := \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(\mathsf{C} + \varepsilon \mathsf{K}) - \mathrm{Vol}_d(\mathsf{C})}{\varepsilon}.$$

(In class, we will show that this limit exists.) Draw some pictures. Use the Brunn–Minkowski inequality to prove Minkowski's first inequality

$$S_{d-1}(\mathsf{C}; \mathsf{K}) \geq \mathrm{Vol}_d(\mathsf{C})^{(d-1)/d} \cdot \mathrm{Vol}_d(\mathsf{K})^{1/d}.$$

**Hint:** Let $f(\tau) := \mathrm{Vol}_d((1-\tau)\mathsf{C} + \tau \mathsf{K})^{1/d}$. The result is equivalent to $f'(0) \geq f(1) - f(0)$.

c) Show that the $\mathsf{K}$-surface area of the Euclidean ball is proportional to the mean width of $\mathsf{K}$:

$$d\, S_{d-1}(\mathsf{B}_d; \mathsf{K}) = \kappa_{d-1}\, V_1(\mathsf{K}).$$

d) Derive Urysohn's mean-width inequality and give an interpretation in words:

$$\frac{V_1(\mathsf{K})}{V_1(\mathsf{B}_d)} \geq \left( \frac{\mathrm{Vol}_d(\mathsf{K})}{\mathrm{Vol}_d(\mathsf{B}_d)} \right)^{1/d}.$$

e) (**) Prove the co-area formula. For a smooth, compactly supported function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} \|\nabla f(\boldsymbol{x})\|_2 \, \mathrm{d}\boldsymbol{x} = \int_0^\infty \mathrm{d}t\, S_{d-1}(\{|f(\boldsymbol{x})| \geq t\}).$$

As usual, $S_{d-1}$ is the Minkowski surface area. Draw a picture. **Hint:** In $\mathbb{R}^2$, this can be accomplished by using the arclength parameterization of the boundary of the level set.

f) Show that isoperimetric inequality implies the following (optimal) Sobolev inequality. For a smooth, compactly supported function $f : \mathbb{R}^d \to \mathbb{R}$ and $p = d/(d-1)$,

$$\left( \frac{1}{\kappa_d} \int_{\mathbb{R}^d} |f(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p} \leq \frac{1}{\omega_d} \int_{\mathbb{R}^d} \|\nabla f(\boldsymbol{x})\|_2 \, \mathrm{d}\boldsymbol{x}.$$

(*) Prove that the converse holds in $\mathbb{R}^2$; that is, Sobolev implies isoperimetry.

3. **(Steiner Symmetrization).** Steiner symmetrization is a very useful tool for establishing geometric results. We already saw a proof of the isoperimetric inequality using this technique, but we can do other things besides.

a) Let $\mathsf{C}, \mathsf{K} \subset \mathbb{R}^d$ be nonempty convex bodies. Use the Gross sphericity theorem to construct a single sequence $\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3, \dots$ of hyperplanes for which

$$\left.\begin{array}{l} \mathrm{st}_{\mathsf{H}_n} \mathrm{st}_{\mathsf{H}_{n-1}} \dots \mathrm{st}_{\mathsf{H}_1} \mathsf{C} \to \mathrm{const}(\mathsf{C})\, \mathsf{B}_d \\ \mathrm{st}_{\mathsf{H}_n} \mathrm{st}_{\mathsf{H}_{n-1}} \dots \mathrm{st}_{\mathsf{H}_1} \mathsf{K} \to \mathrm{const}(\mathsf{K})\, \mathsf{B}_d \end{array}\right\} \quad \text{as } n \to \infty.$$

What are the constants? **Hint:** Alternate.

b) Use Steiner symmetrization and part (a) to establish Brunn–Minkowski in the form

$$\mathrm{Vol}_d(\mathsf{C} + \mathsf{K})^{1/d} \geq \mathrm{Vol}_d(\mathsf{C})^{1/d} + \mathrm{Vol}_d(\mathsf{K})^{1/d}$$

c) Establish the Meyer–Pajor lemma: For a norm ball $\mathsf{K} \subset \mathbb{R}^d$ and a hyperplane $\mathsf{H}$,

$$\mathrm{Vol}_d^{\circ}(\mathrm{st}_{\mathsf{H}} \mathsf{K}) \geq \mathrm{Vol}_d^{\circ}(\mathsf{K}).$$

**Hints:** Assume that $\mathsf{H} = \{\boldsymbol{x} \in \mathbb{R}^d : x_d = 0\}$. Consider the slicing operation $\mathsf{A}(t) = \{\boldsymbol{x} \in \mathbb{R}^{d-1} : (\boldsymbol{x}, t) \in \mathsf{A}\}$. Confirm that $\frac{1}{2}(\mathsf{K}^{\circ}(t) + \mathsf{K}^{\circ}(-t)) \subset (\mathrm{st}_{\mathsf{H}} \mathsf{K})^{\circ}(t)$. Use symmetry of $\mathsf{K}^{\circ}$ and the Brunn slicing theorem.

d) Deduce the Blaschke–Santaló inequality: For a norm ball $\mathsf{K} \subset \mathbb{R}^d$,

$$\mathrm{Vol}_d(\mathsf{K}) \cdot \mathrm{Vol}_d^{\circ}(\mathsf{K}) \leq \mathrm{Vol}_d(\mathsf{B}_d)^2.$$

That is, the Euclidean ball maximizes the product of the volume and the dual volume.

e) (*) Prove Mahler's inequality: For a norm ball $\mathsf{K} \subset \mathbb{R}^d$,

$$\frac{4^d}{(d!)^2} \leq \mathrm{Vol}_d(\mathsf{K}) \cdot \mathrm{Vol}_d^{\circ}(\mathsf{K}).$$

The conjectured minimizer is the $\ell_1^d$ ball, which suggests the bound is far from sharp. (**) Bourgain & Milman have established a lower bound of $(\mathrm{c}/d)^d$ for an absolute constant. Can you prove this?

4. **(*John's Ellipsoids).** The characterization of maximum-volume ellipsoids extends from norm balls to general convex bodies. Let $\mathsf{C} \subset \mathbb{R}^d$ be a convex body with $\dim \mathsf{C} = d$.

a) Prove that $\mathsf{C}$ contains a unique ellipsoid $\mathsf{E}$ with maximum volume.

b) Prove that there is an affine transformation $\boldsymbol{T}$ for which $\mathsf{B}_d$ is the maximum-volume ellipsoid of the set $\boldsymbol{T}\mathsf{C}$. We say that $\boldsymbol{T}\mathsf{C}$ is in *John's position*.

c) Suppose that $\mathsf{B}_d \subset \mathsf{C}$. Show that the following are equivalent:

- $\mathsf{B}_d$ is the maximum-volume ellipsoid of $\mathsf{C}$.
- There are contact points $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m \in \mathsf{B}_d \cap \mathrm{bd}\, \mathsf{C}$ and weights $\alpha_1, \dots, \alpha_m > 0$ with

$$\sum\nolimits_{i=1}^m \alpha_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i = \mathbf{I}_d \quad \text{and} \quad \sum\nolimits_{i=1}^m \alpha_i \boldsymbol{u}_i = \mathbf{0}.$$

d) Let $\mathsf{E}$ be the maximum-volume ellipsoid of the set $\mathsf{C}$. Prove that $\mathsf{E} \subset \mathsf{C} \subset d\,\mathsf{E}$. Show that the upper bound can be improved to $\mathsf{C} \subset \sqrt{d}\,\mathsf{E}$ when $\mathsf{C}$ is origin-symmetric.

e) Dualize (a)–(c) to obtain results about the minimum-volume ellipsoid containing $\mathsf{C}$.

f) Let $\Delta_d := \{\boldsymbol{x} \in \mathbb{R}_+^d : \sum_{i=1}^d x_i = 1\}$ be the probability simplex. Find the maximum- and minimum-volume ellipsoids of $\Delta_d$.

g) (**) Prove that the simplex $\Delta_d$ has maximum volume among all convex bodies with the same maximum-volume ellipsoid. Deduce that the simplex solves the reverse isoperimetric problem over all convex bodies. **Hint:** Use Brascamp–Lieb with $f_i(t) = \mathrm{e}^{-t}\mathbb{1}_{\{t\geq 0\}}$.

h) Let $\boldsymbol{\Delta}_d := \{\boldsymbol{A} \in \mathbb{H}_d : \operatorname{trace} \boldsymbol{A} = 1 \text{ and } \boldsymbol{A} \text{ is psd}\}$, where $\mathbb{H}_d$ is the set of $d \times d$ symmetric matrices. Show that the maximum-volume ellipsoid of $\boldsymbol{\Delta}_d$ is a Euclidean ball centered at $d^{-1}\mathbf{I}_d$ with radius $(d(d-1))^{-1/2}$. Show that the minimum-volume ellipsoid of $\boldsymbol{\Delta}_d$ is a Euclidean ball centered at $d^{-1}\mathbf{I}_d$ with radius $(1-1/d)^{1/2}$.

5. **(*Brascamp & Lieb).** This problem contains some topics around the Brascamp–Lieb inequality.

a) Show that every $k$-dimensional section of the unit cube $\mathsf{C} := [-1/2, 1/2]^d$ has volume at most $(d/k)^{k/2}$. **Hint:** Write the section as $\mathsf{C} \cap \mathsf{L}$ for a subspace $\mathsf{L}$ containing the origin. Express $\mathsf{C} \cap \mathsf{L}$ in terms of the vectors $\boldsymbol{P}\mathbf{e}_i$, where $\boldsymbol{P}$ is the orthogonal projector onto $\mathsf{L}$.

b) Ball conjectured a reversed form of Brascamp–Lieb, which was established by Barthe. The proof is very similar to the one we saw in class. Suppose that $\boldsymbol{u}_i \in \mathbb{R}^d$ are unit vectors and $\alpha_i > 0$ satisfy $\sum_{i=1}^m \alpha_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i = \mathbf{I}_d$. Suppose that

$$h(\boldsymbol{y}) \geq \prod\nolimits_{i=1}^m f_i(\theta_i)^{\alpha_i} \quad \text{for all} \quad \boldsymbol{y} = \sum\nolimits_{i=1}^m \theta_i \alpha_i \boldsymbol{u}_i.$$

Then

$$\int_{\mathbb{R}^d} h(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \geq \prod\nolimits_{i=1}^m \left( \int_{\mathbb{R}} f_i(t)\,\mathrm{d}t \right)^{\alpha_i}.$$

Derive the Prékopa–Leindler inequality as a consequence.

c) Suppose that a symmetric convex body $\mathsf{K} \subset \mathbb{R}^d$ has the Euclidean ball $\mathsf{B}_d$ as its minimum-volume enclosing ellipsoid. Prove that the minimum volume is achieved by a scaled $\ell_1^d$ ball. **Hint:** Use the representation of volume that is dual to the formula in problem (1)(d).

d) (**) Lieb obtained a powerful extension of Brascamp–Lieb. Here is a formulation due to Ball. Suppose that $\boldsymbol{P}_i : \mathbb{R}^d \to \mathbb{R}^d$ are orthogonal projectors and $\alpha_i > 0$ are such that $\sum_{i=1}^m \alpha_i \boldsymbol{P}_i = \mathbf{I}_d$. For nonnegative, integrable functions $f_i : \operatorname{ran}(\boldsymbol{P}_i) \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} \prod\nolimits_{i=1}^m f_i(\boldsymbol{P}_i \boldsymbol{x})^{\alpha_i}\,\mathrm{d}\boldsymbol{x} \leq \prod\nolimits_{i=1}^m \left( \int_{\operatorname{ran}(\boldsymbol{P}_i)} f_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right)^{\alpha_i}.$$

Equal Gaussians saturate the inequality. Barthe invented a proof similar to the argument in class, but it requires more sophisticated methods from optimal transport.

e) (**) Formulate a general statement of the reverse Brascamp–Lieb inequality by combining parts (b) and (d). Prove it.

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